A.8. Some special functions and graphing techniques.

Polynomial function. A **polynomial function** of degree n is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 \qquad (a_n \neq 0)$$

where a_n, a_{n-1},...,a_1, a_0 are constants and n is a nonnegative integer. The domain is all real numbers. The coefficient a_n is called the **leading coefficient**.

Examples of polynomial functions:

$n = 0$: $f(x) = a_0$ constant

$n = 1$: $f(x) = a_1 x + a_0$ linear

$n = 2$: $f(x) = a_2 x^2 + a_1 x + a_0$ quadratic

Example 1. Which of the following are polynomials? For each polynomial determine its degree and circle its leading coefficient.

1. $f(x) = x^5 + x^2 - 6 - x^{11}$ 11th degree
2. $g(x) = 5\sqrt{x} + 4x - 2 = 5x^{1/2} + 4x - 2$ not a polynomial
3. $h(x) = 2x^4 - x^2 + x$ 4th degree
4. $r(x) = x^2 + x^{-2}$ not a polynomial

Rational function. A **rational function** has the form

$$R(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + ... + b_1 x + b_0} = \frac{\text{polynomial}}{\text{polynomial}}$$

The domain of $R(x)$ is all real numbers for which $b_m x^m + b_{m-1}x^{m-1} + ... + b_1 x + b_0 \neq 0$.

Example 2. Find the domain of the following functions.

1. $f(x) = \dfrac{x+1}{x+3}$

$x + 3 \neq 0$

$x \neq -3$

Domain: $\boxed{(-\infty, -3); (-3, \infty)}$

2. $g(x) = \dfrac{x+1}{x^2 - 4}$

$x^2 - 4 \neq 0$

$(x-2)(x+2) \neq 0$

$x \neq 2$, $x \neq -2$

Domain: $\boxed{(-\infty, -2) \cup (-2, 2) \cup (2, \infty)}$

3. $f(x) = \dfrac{x^3 + 1}{x^2 - x - 6}$

$x^2 - x - 6 \neq 0$

$(x+2)(x-3) \neq 0$

$x \neq -2$, $x \neq 3$

Domain: $\boxed{(-\infty, -2) \cup (-2, 3) \cup (3, \infty)}$

Power function. A **power function** is a function of the form

$$f(x) = kx^r$$

where k and r are any real numbers.
Examples of power functions.

1. $r = n$, n is a positive integer. Then $f(x)$ is a polynomial with only one term.

   $y = x,\ y = x^2,\ y = x^3, \ldots$

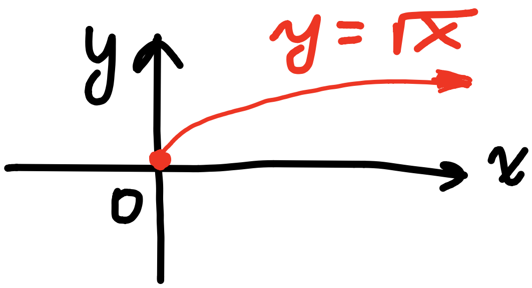
   $y = 5x,\ y = \frac{1}{2}x^2, \ldots$

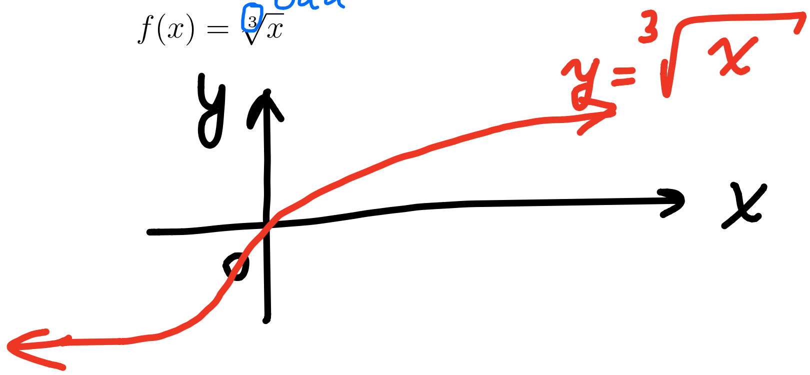
   Domain is $\boxed{(-\infty, \infty)}$

2. $r = \dfrac{1}{n}$, n is a positive integer. Then $f(x) = x^{1/n} = \sqrt[n]{x}$ is a **root function**.

   even $\qquad\qquad\qquad\qquad$ odd

   $f(x) = \sqrt[2]{x}$ $\qquad\qquad\qquad\qquad$ $f(x) = \sqrt[3]{x}$

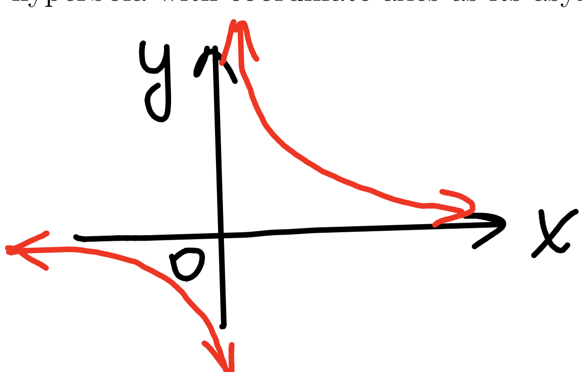




   Domain is $\boxed{[0, \infty)}$ $\qquad\qquad$ Domain is $\boxed{(-\infty, \infty)}$

3. $r = -1$. Then $f(x) = x^{-1} = \dfrac{1}{x}$ is a **reciprocal function**.

   Its graph is a hyperbola with coordinate axes as its asymptotes.



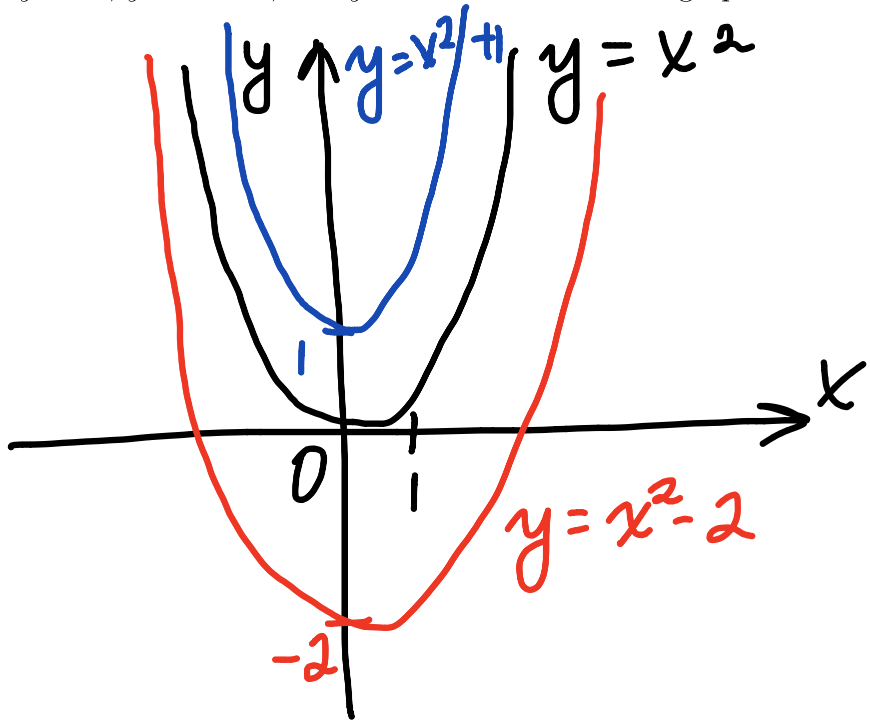
   Domain is $\boxed{(-\infty, 0) \cup (0, \infty)}$

NOTE: your calculator may graph power functions incorrectly when $r = \dfrac{m}{n}$. Instead of entering $y = x^{m/n}$, try $y = (x^m)^{1/n}$ or $y = (x^{1/n})^m$

Graphing technics

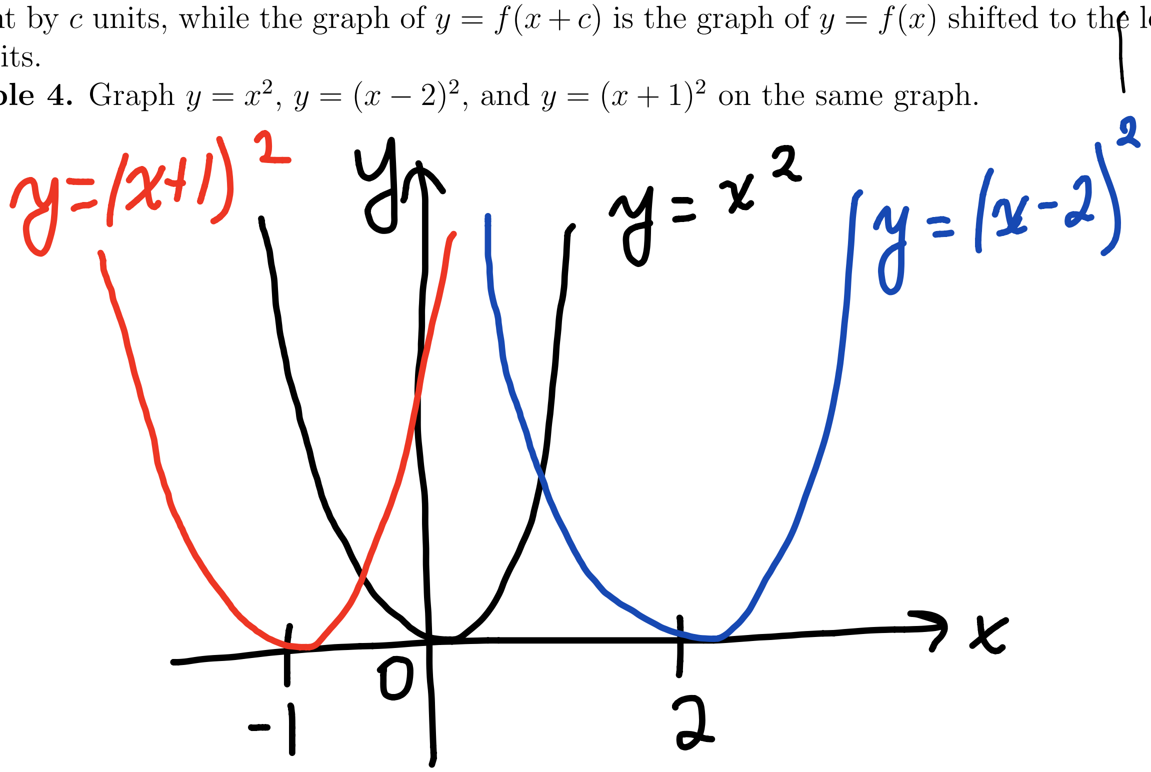
Vertical shift. For $k > 0$, the graph of $y = f(x) + k$ is the graph of $y = f(x)$ shifted upward by k units, while the graph of $y = f(x) - k$ is the graph of $y = f(x)$ shifted downward by k units.

Example 3. Graph $y = x^2$, $y = x^2 + 1$, and $y = x^2 - 2$ on the same graph.

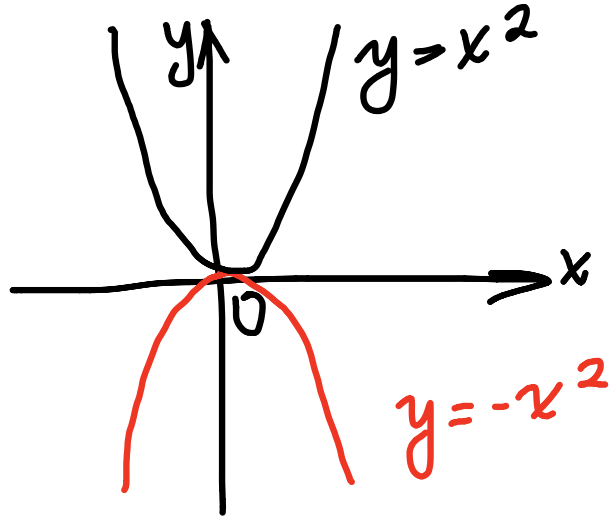


Horizontal shift. For $c > 0$, the graph of $y = f(x - c)$ is the graph of $y = f(x)$ shifted to the right by c units, while the graph of $y = f(x + c)$ is the graph of $y = f(x)$ shifted to the left by c units.

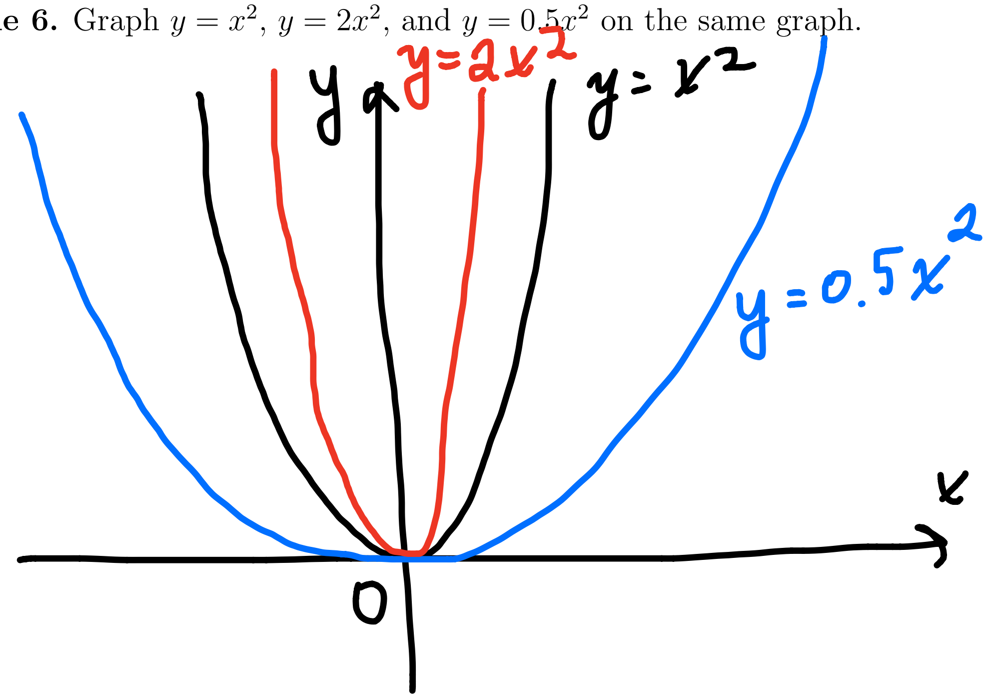
Example 4. Graph $y = x^2$, $y = (x - 2)^2$, and $y = (x + 1)^2$ on the same graph.



Reflection. The graph of $y = -f(x)$ is the reflection of the graph $y = f(x)$ in the x-axis.
Example 5. Graph $y = x^2$ and $y = -x^2$ on the same graph.

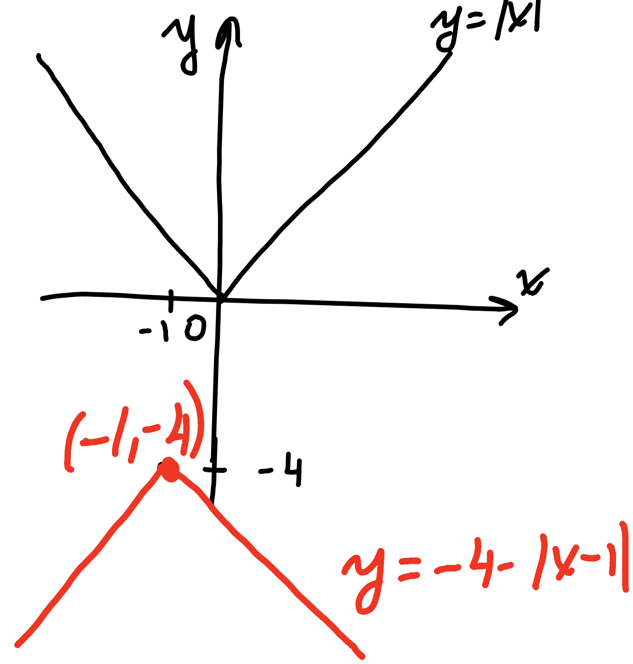


Expansion and Contraction. For $a > 0$, the graph of $y = af(x)$ is an expansion (vertically) of the graph of $y = f(x)$ if $a > 1$, and a contraction (vertically) of the graph of $y = f(x)$ if $0 < a < 1$.
Example 6. Graph $y = x^2$, $y = 2x^2$, and $y = 0.5x^2$ on the same graph.



Example 7. How is the graph of $y = -4 - |x+1|$ found from the graph of $y = |x|$?

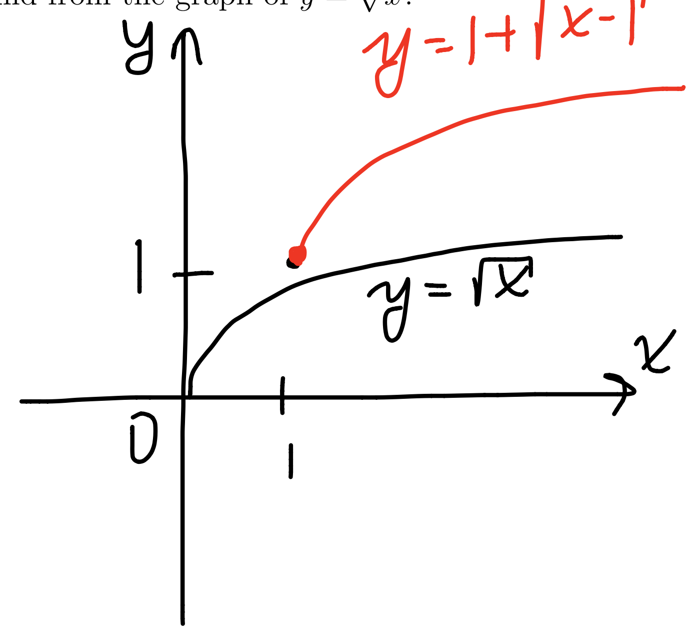
- move left 1 unit
- reflect over x-axis
- move down 4 units



Example 8. How is the graph of $y = 1 + \sqrt{x-1}$ found from the graph of $y = \sqrt{x}$?

- move right 1unit
- move up 1 unit

Domain : $\boxed{[1, \infty)}$



NOTE: the order of transformations is important. Perform the transformations in the order you would perform the mathematical calculations for each value of x when determining the resulting y-value.