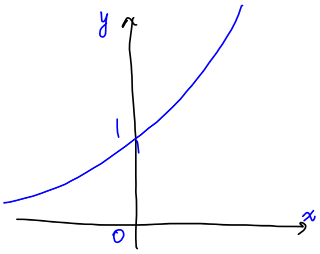
1.3 **Exponential functions.**

Definition The equation

$$f(x) = b^x, \quad b > 0, b \neq 1$$

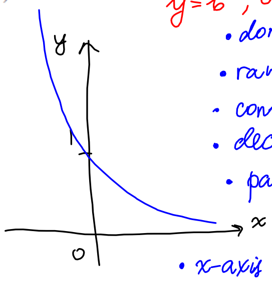
defines an **exponential function** for each different constant b, called the **base**. The **domain** of the function is $(-\infty, \infty)$ and the **range** is $(0, \infty)$.

$y = b^x,\ b > 1$



- domain $(-\infty, \infty)$
- range $(0, \infty)$
- continuous for all x
- increasing for all x
- passes through $(0,1)$
- x-axis is the horizontal asymptote as x goes to $-\infty$

$y = b^x,\ 0 < b < 1$



- domain $(-\infty, \infty)$
- range $(0, \infty)$
- continuous for all x
- decreasing for all x
- passes through $(0,1)$
- x-axis is the horizontal asymptote as x goes to ∞

Basic properties of the graph of $f(x) = b^x$**,** $b > 0$**,** $b \neq 1$**.**

1. All graphs will pass through the point (0,1).
2. All graphs are continuous curves, with no holes or jumps.
3. The x axis is horizontal asymptote.
4. If $b > 1$, then b^x increases as x increases.
5. If $0 < b < 1$, then b^x decreases as x increases.

Properties of exponential functions

For a and b positive, $a \neq 1$, $b \neq 1$, and x and y real,

1. Exponent laws:

$$\boxed{a^{x+y} = a^x a^y} \quad \boxed{a^{x-y} = \frac{a^x}{a^y}} \quad \boxed{(a^x)^y = a^{xy}} \quad \boxed{(ab)^x = a^x b^x}$$

2. $a^x = a^y$ if and only if $x = y$

3. For $x \neq 0$,

$$a^x = b^x \text{ if and only if } a = b$$

Example 1. Simplify each expression:

a) $(2^x)^y$, $= 2^{xy}$

b) $\dfrac{6^{x+5}}{2^{x-2}}$, $= \dfrac{(3\cdot 2)^{x+5}}{2^{x-2}} = \dfrac{3^{x+5}\cdot 2^{x+5}}{2^{x-2}} = 3^{x+5}\,\dfrac{2^{x+5}}{2^{x-2}} = 3^{x+5}\,2^{x+5-(x-2)}$

$= 3^{x+5}\,2^{7} = 3^{x}\cdot 3^{5}\cdot 2^{7} = \boxed{(31104)(3^{x})}$

1

c) $\dfrac{3^{2x-1}}{3^{x-2}}$, $= 3^{2x-1-(x-2)} = 3^{x+1} = \boxed{3\cdot 3^{x}}$

d) $2^{x-1}4^{4-x} = 2^{x-1}(2^2)^{4-x} = 2^{x-1}\,2^{2(4-x)} = 2^{x-1}\cdot 2^{8-2x} = 2^{x-1+8-2x}$

$= 2^{7-x} = 2^{7}2^{-x} = \boxed{128\cdot 2^{-x}}$

Example 2. Solve each equation for x.

a) $\cancel{4}^{5x-x^2} = \cancel{4}^{-6}$,

$5x - x^2 = -6$

$x^2 - 5x - 6 = 0$

$(x+1)(x-6) = 0$

$\boxed{x_1 = -1 \quad x_2 = 6}$

b) $5^{\cancel{3}} = (x+2)^{\cancel{3}}$.

$5 = x + 2$

$\boxed{x = 3}$

Base e exponential function.

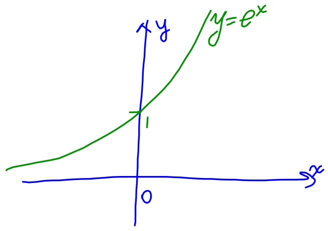
Number e is irrational, it can be approximated as closely as we like by evaluating the expression

$$\left(1+\frac{1}{x}\right)^x$$

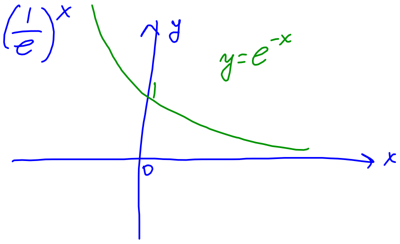
for sufficiently large x.

$$\boxed{e \approx 2.718\ 281\ 828\ 459}$$

Exponential function with base e. Exponential functions with base e and base $1/e$, respectively, are defined by



$$y = e^x \quad \text{and} \quad y = e^{-x}. = \left(\frac{1}{e}\right)^x$$



The **domain** of both of these functions is $(-\infty, \infty)$ and the **range** is $(0, \infty)$.

Growth and decay applications.

Functions of the form $y = ce^{kt}$, where c and k are constants and the independent variable t represents time, are often used to model population growth and radioactive decay. Note, that if $t = 0$, then $y = c$. So, the constant c represents the initial population (or initial amount). The constant k is called the **relative growth rate**.

We say that **populations is growing continuously at relative growth rate** k to mean that the population y is given by the model $\boxed{y = ce^{kt}}$.

c is the initial population
k is the relative growth rate

Example 3. In 2006, the estimated population in Ethiopia was 75 millions people with a relative growth rate of 2.3%.

(a) Write an equation that models the population growth in Ethiopia, letting 2006 be year 0.

$$y = ce^{kt}$$
$$C = 75{,}000{,}000$$
$$k = 2.3\% = 0.023$$
$$\boxed{y = 75{,}000{,}000\, e^{0.023t}}$$

(b) Based on the model, what is the expected population in Ethiopia (to the nearest million) in 2015?

$$2015 - 2006 = 9$$
$$y(9) = 75{,}000{,}000\, e^{0.023(9)} \approx 92\,248\,692.88 \approx \boxed{92{,}000{,}000}$$

Compound interest

The free paid to use another's money is called **interest**. It is usually computed as a percent (called **interest rate**) of the principal over a given period of time. If, at the end of a payment period, the interest due is reinvested at the same rate, then the interest earned as well as the principal will earn interest during the next payment period. Interest paid on interest reinvested is called **compound interest**, and may be calculated using the following compound interest formula:

If a **principal** P **(present value)** is invested at an annual **rate** r (expressed as a decimal) compounded m times a year, then the **amount** A **(future value)** in the account at the end of t years is given by

$$\boxed{A = P\left(1 + \frac{r}{m}\right)^{mt}.}$$

A - future value
P - present value
r - rate
m is the # of compunds

Note: P could be replaced by A_0, but convention dictates otherwise. For given r and m, the amount A is equal to the principal P multiplied by the exponential function b^t, where $b = (1 + r/m)^m$.

Example 4. A couple just had a baby. How much should they invest now at 5.5% compounded monthly in order to have \$ 40,000 for the child's education 17 years from now? Compute the answer to nearest dollar.

\$15,737

We are going to use **TVM Solver**.

1) Go to FINANCE on your calculator (press $\boxed{\text{APPS}}$ button and select option $\boxed{1}$).

2) Select $\boxed{1}$ (TVM Solver). You will have the following on your screen:

N= "*the total number of times compounded = (number of years)× (number of time compounded per year)*"

I%= "*interest rate*"

PV= "*present value*"

PMT=0 "*we are not making payments* "

FV= "*future value*"

P/Y= "*number of time compounded per year* "

C/Y= "*number of time compounded per year* "

3) Move your cursor to the value you are soving for and hit $\boxed{\text{ALPHA}}$ and then $\boxed{\text{ENTER}}$.

A is the future value
P is the present value
r is the rate in decimal

Continuous Compound Interest: $\boxed{A = Pe^{rt}}$

Example 5. Suppose \$29,000 is deposited into an account paying 7.5% annual interest. How much will be in the account after 5 years if the account is compounded continuously?

$A = Pe^{rt}$

$P = 29000$

$r = 7.5\% = 0.075$

$t = 5$

$A = 29000e^{0.075(5)} \approx \boxed{\$\,42194.75}$

Effective Rate of Interest (Effective Annual Yield) The simple interest rate that would produce the same accumulated amount in one year as the nominal rate compounded m times a year.

$$\boxed{r_{\text{eff}} = \left(1 + \frac{r}{m}\right)^m - 1}$$

The effective interest rate is often used when comparing two accounts that are compounded differently.

On the calculator...

1. Go to FINANCE and select EFF.
2. Give the arguments as follows: EFF(r, m) where r is given in % form

Example 6. What is the effective annual yield on an account paying 6% interest per year, compounded monthly?

$\boxed{6.17\%}$

Example 7. Of the two options below,

A: 8% compounded semi-annually   8.16%

B: 7.9% compounded daily   8.22%

4

(a) Which is the better investment?   B

(b) Which is the better credit card rate?   A

Effective Rate of Interest for Continuously Compounded Interest $\boxed{r_{\text{eff}} = e^r - 1}$

Example 8. What is the effective annual yield on an account paying 6% interest per year, compounded continuously?

$r_{eff} = e^r - 1, \quad r = 6\% = 0.06$

$r_{eff} = e^{0.06} - 1 = \boxed{0.062 = 6.2\%}$