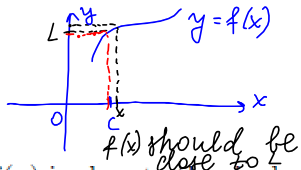
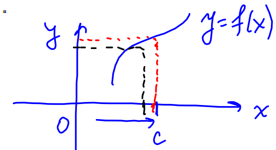
Section 3.1 **Limits**

Definition We write



$$\lim_{x\to c} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to c$$

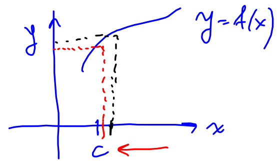
and say "the limit of $f(x)$, as x approaches c, if the functional value $f(x)$ is close to the single number L whenever x is close, but not equal, to c (on both sides of c).



Definition We write

$$\lim_{x\to c^-} f(x) = K$$

and call K the **limit from the left** or the **left-hand limit** if $f(x)$ is close to K whenever x is close to, but to the left of, c on the real number line.



We write

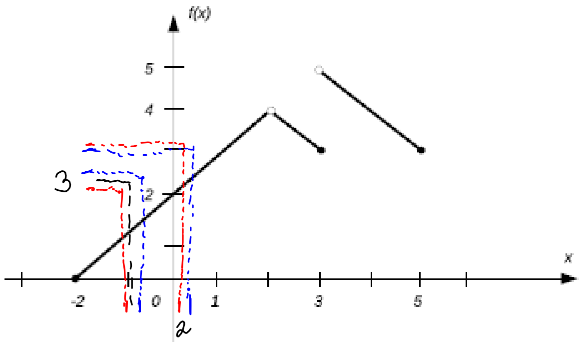
$$\lim_{x\to c^+} f(x) = L$$

and call L the **limit from the right** or the **right-hand limit** if $f(x)$ is close to L whenever x is close to, but to the right of, c on the real number line.

If no direction is specified in a limit statement, we will always assume that the limit is **two-sided** or **unrestricted**.

$$\lim_{x\to c} f(x) = L \text{ if and only if } \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$$

Example 1. Given the graph of the function f



$f(1)=3$

Find:

1. $\lim_{x\to 1} f(x) = 3$
2. $\lim_{x\to 2^+} f(x) = 4$
3. $\lim_{x\to 2^-} f(x) = 4$

$\lim_{x\to 2} f(x) = 4$ although $f(2)$ is not defined

4. $\lim_{x\to 3^+} f(x) = 5$

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$\lim_{x\to 3} f(x)$ DNE.

5. $\lim_{x\to 3^-} f(x) = 3$

Limit laws Suppose that k is a constant and the limits $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. Then

1. $\lim_{x\to c}[f(x)+g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
2. $\lim_{x\to c}[f(x)-g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$
3. $\lim_{x\to c} kf(x) = k\lim_{x\to c} f(x)$
4. $\lim_{x\to c} f(x)g(x) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$
5. $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ if $\lim_{x\to c} g(x) \neq 0$
6. $\lim_{x\to c}[f(x)]^n = \left[\lim_{x\to c} f(x)\right]^n$ where n is a positive integer
7. $\boxed{\lim_{x\to c} k = k}$ 8. $\boxed{\lim_{x\to c} x = c}$
9. $\lim_{x\to c} x^n = c^n$ where n is a positive integer
10. $\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$ where n is a positive integer
11. $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$ where n is a positive integer

Example 2. Find each limit if it exists.

1. $\lim_{x\to 5}(3x+6) = \lim_{x\to 5}(3x) + \lim_{x\to 5} 6$

$= \lim_{x\to 5} x + \lim_{x\to 5} 6$

$= 3(5) + 6$

$= \boxed{21}$

shortcut:

$\lim_{x\to 5}(3x+6) = 3(5)+6$

$= \boxed{21}$

2. $\lim_{x\to 1} \frac{x^2+2x-3}{x+1} = \frac{1+2-3}{2} = \boxed{0}$

3. $\lim_{x\to 2} x(x+1) \overset{\text{plug } x=2}{=\!=\!=} 2(2+1)$

$= \boxed{6}$

4. $\lim_{x\to -1} \sqrt{4-5x} = \sqrt{4-5(-1)}$

$= \sqrt{4+9}$

$= \boxed{3}$

Limits of polynomial and rational functions.

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1. $\boxed{\lim_{x\to c} f(x) = f(c)}$ if f is a polynomial function

2. $\boxed{\lim_{x\to c} r(x) = r(c)}$ if r is a rational function with a nonzero denominator at $x = c$.

Example 3. Find each limit if it exists.

1. $\lim_{x\to 1}(x^5 - 3x^3 + 5x + 4) \overset{\text{plug } x=1}{=} 1-3+5+4$
$= \boxed{7}$

2. $\lim_{x\to 3} \dfrac{x^2+2}{x^3+4} = \dfrac{3^2+2}{3^3+4} = \boxed{\dfrac{11}{31}}$

3. $\lim_{x\to 2} \dfrac{x|x-2|}{x-2}$ DNE

$|x-2| = \begin{cases} x-2, & \text{if } x-2 \geq 0 \\ -(x-2), & \text{if } x-2 < 0 \end{cases}$

one-sided limits.

$\lim_{\substack{x\to 2^+ \\ x>2}} \dfrac{x|x-2|}{x-2} = \lim_{x\to 2^+} \dfrac{x\cancel{(x-2)}}{\cancel{x-2}} = \lim_{x\to 2^+} x = \boxed{2}$

$\neq$

$\lim_{\substack{x\to 2^- \\ x<2}} \dfrac{x|x-2|}{x-2} = \lim_{x\to 2^-} \dfrac{x(-\cancel{(x-2)})}{\cancel{x-2}} = \lim_{x\to 2^-} (-x) = \boxed{-2}$

Indeterminate form. If $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c} \frac{f(x)}{g(x)}$ is said to be **indeterminate** , or, more specifically, a 0/0 indeterminate form.

If you try finding $f(a)$ and you end up with 0/0 indeterminate form, then

1. Factor both the numerator and denominator, fully.
2. Cancel all common factors.
3. Substitute x = a into the simplified function.

Example 3. Let $f(x) = \frac{x^2 + x - 6}{x^2 + 2x - 3}$. Find

1. $\lim_{x\to -3} f(x), = \lim_{x\to -3} \frac{x^2+x-6}{x^2+2x-3}$ $\quad \left| \frac{(-3)^2-3-6}{(-3)^2+2(-3)-3} = \frac{0}{0} \right|$

$$= \lim_{x\to -3} \frac{\cancel{(x+3)}(x-2)}{\cancel{(x+3)}(x-1)}$$

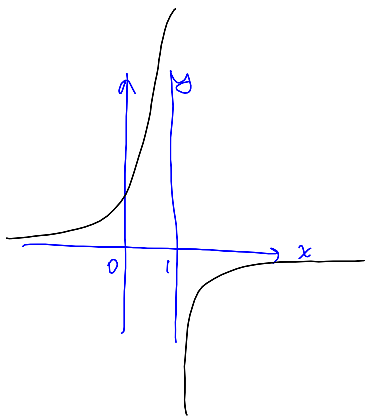
$$= \lim_{x\to -3} \frac{x-2}{x-1} = \frac{-3-2}{-3-1} = \boxed{\frac{5}{4}}$$

2. $\lim_{x\to 1} f(x) = \lim \frac{x^2+x-6}{x^2+2x-3}$ $\boxed{\text{DNE}}$ $\left| \frac{1+1-6}{1+2-3} = \frac{-4}{0} \right|$

$$\lim_{x\to 1^-} \frac{x^2+x-6}{x^2+2x-3} = \infty$$

$$\neq$$

$$\lim_{x\to 1^+} \frac{x^2+x-6}{x^2+2x-3} = -\infty$$



$x^2+2x-3 = (x+3)(x-1)$

why $x=-3$ is not a vertical asymptote?

Plug $x=-3$ into the numerator:

x^2+x-6: $(-3)^2+(-3)-6=0$.

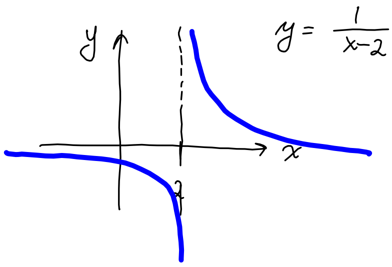
Infinite limits and limits at infinity.

If a function f either increases or decreases without bound as x approaches a, we say that the

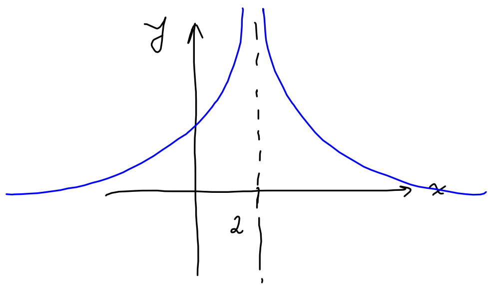
$$\boxed{\lim_{x\to a} f(x) = \infty.}$$ ∞ or $-\infty$

Example 4. Find each of the following limits

1. $\lim_{x\to 2} \frac{1}{x-2}$, DNE $\left| \frac{1}{2-2} = \frac{1}{0} \right|$



2. $\lim_{x\to 2} \frac{1}{(x-2)^2}$. $\boxed{\infty}$ $\left| \frac{1}{(2-2)^2} = \frac{1}{0} \right|$



The vertical line $x = a$ is a vertical asymptote for the graph of $y = f(x)$ if

$$\boxed{f(x) \to \infty \text{ or } f(x) \to -\infty \text{ as } x \to a^+ \text{ or } x \to a^-.}$$

If $f(x) = n(x)/d(x)$ is a rational function, $d(c) = 0$ and $n(c) \neq 0$, then the line $x = c$ is a vertical asymptote of the graph of f.

Example 5. Find vertical asymptotes of the function $\dfrac{3x+2}{x^3+x^2-2x}$.

1) set the denominator $= 0$

$x^3+x^2-2x=0$

$x(x^2+x-2)=0$

$x(x+2)(x-1)=0$

$x_1=0,\ x_2=-2,\ x_3=1$

2) Plug $x_1=0,\ x_2=-2,\ x_3=1$ into the numerator

$3(0)+2=2\neq 0$

$3(-2)+2=-4\neq 0$

$3(1)+2=5\neq 0$

$x=0$

$x=-2$

$x=1$

vertical asymptotes

The symbol ∞ also can be used to indicate that an independent variable is increasing or decreasing without bound. We write $x \to \infty$ to indicate that x increases without bound through positive values and $x \to -\infty$ to indicate that x decreases without bound through negative values.

A line $y = b$ is a horizontal asymptote of the graph of $y = f(x)$ if either $\lim\limits_{x\to\infty} f(x) = b$ or $\lim\limits_{x\to-\infty} f(x) = b$.

Limits of power functions at infinity. If p is a positive real number and k is any real constant, then

1. $\lim_{x\to-\infty} \frac{k}{x^p} = 0$ 2. $\lim_{x\to\infty} \frac{k}{x^p} = 0$
3. $\lim_{x\to-\infty} kx^p = \pm\infty$ 4. $\lim_{x\to\infty} kx^p = \pm\infty$

provided that x^p is a real number for negative values for x. The limits in 3 and 4 will be either $-\infty$ or ∞, depending on k and p.

Limits of polynomial functions at infinity. If

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0, \quad a_n \neq 0, \quad n \geq 1$$

then

$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} a_n x^n = \pm\infty$$

and

$$\lim_{x\to-\infty} p(x) = \lim_{x\to-\infty} a_n x^n = \pm\infty.$$

Each limit will be either ∞ or $-\infty$, depending on a_n and n.

Polynomials of degree 1 or greater never have horizontal asymptotes.

Limits of rational functions at infinity.

$$\lim_{x\to\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0} = \lim_{x\to\infty} \frac{a_n x^n}{b_m x^m} =$$

$$\begin{cases} \frac{a_n}{b_m}, & \text{if } n = m \\ 0, & \text{if } n < m \\ \infty, & \text{if } n > m \end{cases}$$

A rational function can have at most one horizontal asymptote.

Limits of power functions at infinity. If p is a positive real number and k is any real constant, then

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5

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Limits of polynomial functions at infinity. If

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0, \quad a_n \neq 0, \quad n \geq 1$$

then

$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} a_n x^n = \pm\infty$$

and

$$\lim_{x\to-\infty} p(x) = \lim_{x\to-\infty} a_n x^n = \pm\infty.$$

Each limit will be either ∞ or $-\infty$, depending on a_n and n.

Polynomials of degree 1 or greater never have horizontal asymptotes.

Limits of rational functions at infinity.

$$\lim_{x\to\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \dots + b_1 x + b_0} = \lim_{x\to\infty} \frac{a_n x^n}{b_m x^m} =$$

$$\begin{cases} \frac{a_n}{b_m}, & \text{if } n = m \\ 0, & \text{if } n < m \\ \infty, & \text{if } n > m \end{cases}$$

A rational function can have at most one horizontal asymptote.

Example 6. Evaluate the limit.

1. $\lim\limits_{x\to\infty} \frac{\boxed{7}x^3+4x}{\boxed{2}x^3-x^2+3} = \boxed{\frac{7}{2}}$

$$\lim_{x\to -\infty} \frac{\boxed{7x^3}+4x}{\boxed{2x^3}-x^2+3} = \lim_{x\to -\infty} \frac{7}{2}\frac{x^3}{x^3} = \lim_{x\to -\infty} \frac{7}{2} = \boxed{\frac{7}{2}}$$

2. $\lim\limits_{x\to\infty} \frac{x+4}{x^3-3} = 0$

3. $\lim_{x \to \infty} \dfrac{x^2 - 3x + 1}{2x + 3} = \infty$

$$\lim_{x \to -\infty} \frac{\boxed{x^2} - 3x + 1}{\boxed{2x} + 3} = \lim_{x \to -\infty} \frac{x^2}{2x} = \lim_{x \to -\infty} \frac{x}{2} = -\infty$$

Continuity.
A function f is continuous at the point $x = c$ if

1. $\lim_{x \to c} f(x)$ exists $\qquad \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$

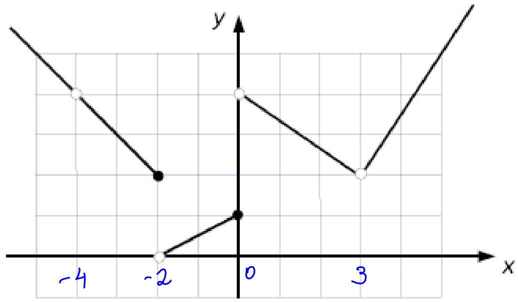
2. $f(c)$ exists

3. $\lim_{x \to c} f(x) = f(c)$.

A function is continuous on the open interval (a, b) if f is continuous at each point on the interval.

If one or more of the three conditions in the definition fails, the function is discontinuous at $x = c$.

Example 7. Discuss the continuity of the function whose graph is



$y = f(x)$

$x = -4$ – hole
$x = -2$ – jump
$x = 0$ – jump
$x = 3$ – hole

points at which $f(x)$ is discontinuous

• Polynomials are continuous on $(-\infty, \infty)$

• Rational functions are discontinuous at zeroes of the denominator.

Example 8. Determine where each function is continuous.

1. $f(x) = 3x + 2$ continuous on $(-\infty, \infty)$

2. $f(x) = x^{25} - x^3 + 4$ continuous on $(-\infty, \infty)$

3. $f(x) = \dfrac{x^2+4}{4-25x^2}$

$4 - 25x^2 = 0$

$25x^2 = 4$

$x^2 = \frac{4}{25}$

$x_1 = -\frac{2}{5}$, $x_2 = \frac{2}{5}$ points of discontinuity

Continuous on $\boxed{(-\infty, -\frac{2}{5}) \cup (-\frac{2}{5}, \frac{2}{5}) \cup (\frac{2}{5}, \infty)}$

4. $f(x) = \begin{cases} x+4, & \text{if } x \le 3 \\ 2x-1, & \text{if } x > 3 \end{cases}$

1) $x+4$ and $2x-1$ are continuous for all x.

2) $f(x)$ is continuous when $x > 3$ and $x < 3$

3) continuity at $x = 3$.

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (2x-1) = 5$

$\#$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x+4) = 7$

$\lim_{x\to 3} f(x)$ DNE

$f(x)$ is discontinuous at $x = 3$.

$f(x)$ is continuous on $\boxed{(-\infty, 3) \cup (3, \infty)}$

5. $f(x) = \dfrac{\cancel{x-1}}{\cancel{(x-1)}(x+1)} = \dfrac{1}{x+1}$

$x+1 = 0 \Rightarrow x = -1$

$f(x)$ is discontinuous at -1.

$f(1) =$ hole

$f(x)$ is discontinuous at 1.

$f(x)$ is continuous on $\boxed{(-\infty, -1) \cup (-1, 1) \cup (1, \infty)}$