Section 3.2 **Rates of change.**

Average rate of change. **Instanteneous rate of change.**

Suppose y is a quantity that depends on another quantity x or $y = f(x)$. If x changes from x_1 to x_2, then the change in x (also called the increment of x) is

$$\boxed{\Delta x = x_2 - x_1}$$

and the corresponding change in y is

$$\boxed{\Delta y = f(x_2) - f(x_1)}$$

The difference quotient

$$\boxed{\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}}$$

is called the average rate of change of y with respect to x over the interval $[x_1, x_2]$.

The instantaneous rate of change of y with respect to x at $x = x_1$ is equal to

$$\boxed{\lim_{\substack{\Delta x \to 0 \\ x_2 \to x_1}} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_2) - f(x_1)}{x_2 - x_1}}$$

Example 1. The population (in thousands) of a city from 1990 to 1996 is given in the following table

year	1990	1991	1992	1993	1994	1995	1996
P (in 1000s)	105	110	117	126	137	150	164

(a) Find the average rate of growth from 1992 to 1996

average rate of growth $= \dfrac{P(1996) - P(1992)}{1996 - 1992} = \dfrac{164 - 117}{4} = 11.75$

The population increased by 11750 people.

(b) Estimate the instantaneous rate of growth in 1993

average rate of growth $1992 \to 1993$: $\dfrac{P(1993) - P(1992)}{1} = \dfrac{126 - 117}{1} = 9$

$\lim_{x \to 1993^-} P(x) = 9$

average rate of growth $1993 - 1994$: $\dfrac{P(1994) - P(1993)}{1} = \dfrac{137 - 126}{1} = 11$

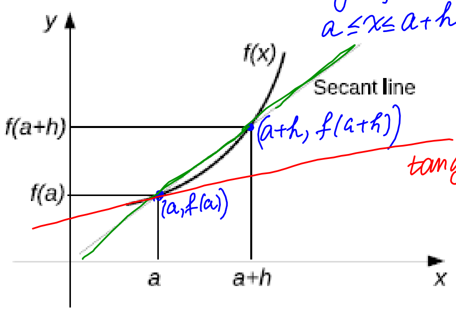
$\lim_{x \to 1993^+} P(x) = 11$

instantaneous rate of growth $= \dfrac{11 + 9}{2} = \boxed{10}$

Slope of a secant line. The tangent line.

slope of the secant line is the average rate of change of f from a to $a+h$

$$\boxed{\text{slope} = \frac{f(a+h)-f(a)}{h}}$$



the slope of the tangent line is the instantaneous rate of change of f at a.

$$\boxed{\text{slope} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)}$$

Given $y = f(x)$, the **slope of the graph** at the point $(a, f(a))$ is given by

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

provided that the limit exists. The slope of the graph is also the **slope of the tangent line** at the point $(a, f(a))$.

Example 2. Find the equation of the tangent line to the curve $y = 2x^2 - 3$ at the point $(2, 5)$.

Equation of the tangent line

$$y - y_0 = m(x - x_0)$$

Passes through $(\underbrace{2}_{x_0}, \underbrace{5}_{y_0})$

$$\text{slope} = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$

$$f(2+h) = 2(2+h)^2 - 3$$
$$= 2(4 + 4h + h^2) - 3$$
$$= 8 + 8h + 2h^2 - 3$$
$$= 5 + 8h + 2h^2$$

$$f(2) = 2(2^2) - 3 = 5$$

$$\text{slope} = \lim_{h\to 0} \frac{5 + 8h + 2h^2 - 5}{h} = \lim_{h\to 0} \frac{8h + 2h^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(8+2h)}{\cancel{h}} = \lim_{h\to 0} (8 + 2h) = \boxed{8}$$

Short cut.

$$\text{slope} = f'(2)$$
$$f(x) = 2x^2 - 3$$
$$f'(x) = 4x$$
$$f'(2) = \boxed{8}$$

Equation of the tangent line:

$$y - 5 = 8(x - 2)$$
$$y - 5 = 8x - 16$$
$$\boxed{y = 8x - 11}$$

Velocity.

Suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement of the object from the origin at time t. Function f is called the **position function** of the object.

$$\boxed{\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{h}}$$

Then the velocity or instantaneous velocity at time $t = a$ is

$$\boxed{v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}} = f'(a)$$

Example 3. The displacement of an object moving in a straight line is given by $s(t) = 1 + 2t + t^2/4$ (t is in seconds).

(a) Find the average velocity over the time period [1,3]

$$\text{average velocity} = \frac{s(3)-s(1)}{3-1} = \frac{\overbrace{1+2(3)+\frac{9}{4}}^{s(3)} - \overbrace{\left(1+2+\frac{1}{4}\right)}^{s(1)}}{2}$$

$$\frac{9}{4}-\frac{1}{4}=\frac{8}{4}=2$$

$$= \frac{\cancel{1}+6+\frac{9}{4}-\cancel{1}-2-\frac{1}{4}}{2} = \frac{6}{2} = \boxed{3}$$

(b) Find the instantaneous velocity when $t = 1$

$$\text{instanteneous velocity} = \lim_{h\to 0} \frac{s(1+h)-s(1)}{h}$$

$$= \lim_{h\to 0} \frac{\left[1+2(1+h)+\frac{(1+h)^2}{4}\right] - \left[1+2+\frac{1}{4}\right]}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{1}+\cancel{2}+2h+\frac{(1+2h+h^2)}{4} - \cancel{1}-\cancel{2}-\frac{1}{4}}{h}$$

$$= \lim_{h\to 0} \frac{2h+\cancel{\frac{1}{4}}+\frac{2h}{4}+\frac{h^2}{4}-\cancel{\frac{1}{4}}}{h}$$

$$2h+\frac{2h}{4} = 2h+\frac{h}{2} = \frac{4h+h}{2} = \frac{5h}{2}$$

$$= \lim_{h\to 0} \frac{\frac{5h}{2}+\frac{h^2}{4}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}\left(\frac{5}{2}+\frac{h}{4}\right)}{\cancel{h}}$$

$$= \lim_{h\to 0} \left(\frac{5}{2}+\frac{h}{4}\right) = \boxed{\frac{5}{2}}$$

shortcut:

instanteneous velocity $= s'(1)$

$s'(t) = 2+\frac{2t}{4} = 2+\frac{t}{2}$

$s'(1) = 2+\frac{1}{2} = \boxed{\frac{5}{2}}$