Section 3.3 **Derivatives**

Definition. The **derivative of a function** f **at a number** a, denoted by $f'(a)$, is

$$\boxed{f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}}$$

if the limit exist.

Example 1. Find $f'(a)$ if $f(x)=\sqrt{x}$, $a>0$.

$[f(x)=\sqrt{x},\ f(a)=\sqrt{a}]$

$(a-b)(a+b)=a^2-b^2$

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{x-a} = \left|\frac{0}{0}\right| = \lim_{x\to a}\frac{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})}{(x-a)(\sqrt{x}+\sqrt{a})}$$

$$=\lim_{x\to a}\frac{(\sqrt{x})^2-(\sqrt{a})^2}{(x-a)(\sqrt{x}+\sqrt{a})} = \lim_{x\to a}\frac{\cancel{x-a}}{\cancel{(x-a)}(\sqrt{x}+\sqrt{a})}$$

$$=\lim_{x\to a}\frac{1}{\sqrt{x}+\sqrt{a}} = \frac{1}{\sqrt{a}+\sqrt{a}} = \boxed{\frac{1}{2\sqrt{a}}}$$

Geometric interpretation of the derivative. $f'(a)$ is the slope of the tangent line to $y=f(x)$ at the point $(a, f(a))$.

Example 2. Find an equation of the tangent line to $f(x)=\sqrt{x}$ at the point where $x=9$.

Equation of the tangent line: $y-\underbrace{y_0}_{3}=m(x-\underbrace{x_0}_{9})$

$\boxed{f'(a)=\frac{1}{2\sqrt{a}} \quad f(x)=\sqrt{x}}$

$y_0=f(9)=\sqrt{9}=3$

Slope of tangent line $m=f'(9)=\frac{1}{2\sqrt{9}}=\frac{1}{6}$

$y-3=\frac{1}{6}(x-9)$

$y=3+\frac{1}{6}x-\frac{9}{6}$

$\boxed{y=\frac{1}{6}x+\frac{3}{2}}$

Other interpretations of the derivative.

- $f'(a)$ is the instanteneous rate of change of $y = f(x)$ with respect to x when $x = a$.

- if $s = f(t)$ is the position function of a particle that moves along a straight line, then $f'(a)$ is the velocity of the particle at time $t = a$

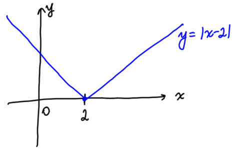
A function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

is called the **derivative of** f.

Definition. A function f is **differentiable at** a if $f'(a)$ exists. It is **differentiable on an open interval** (a,b) if it is differentiable at every number in the interval.

Example 3. Where is the function $f(x) = |x-2|$ differentiable?



$f'(2) - ?$

$$|x-2| = \begin{cases} x-2, & \text{if } x-2 \geq 0 \\ -(x-2), & \text{if } x-2 < 0 \end{cases}$$

$$\lim_{x \to 2^+} \frac{f(x)-f(2)}{x-2} = \lim_{x \to 2^+} \frac{\cancel{x-2}-0}{\cancel{x-2}} = 1$$

$$\lim_{x \to 2^-} \frac{f(x)-f(2)}{x-2} = \lim_{x \to 2^-} \frac{-(x-2)-0}{x-2} = -1$$

$\neq$

$\boxed{f'(2) \text{ DNE}}$

Example 3'. Find the derivative of $f(x)=x^2$ using the limit definition of the derivative.

$f(x) = x^2$
$f(x+h) = (x+h)^2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = \lim_{h \to 0} (2x+h) = \boxed{2x}$$

$\boxed{(x^2)' = 2x}$ $\boxed{(\sqrt{x})' = \frac{1}{2\sqrt{x}}}$

Example 3'. Find the derivative of the function $f(x) = \frac{1}{x}$ using the limit definition of the derivative

$f(x) = \frac{1}{x}$
$f(x+h) = \frac{1}{x+h}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x} - \cancel{x} - h}{h x (x+h)}$$

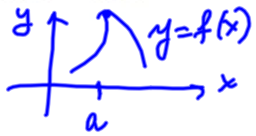
$$= \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h} x (x+\cancel{h}^{0})} = -\frac{1}{x^2}$$

$$\boxed{\left(\frac{1}{x}\right)' = -\frac{1}{x^2}}$$

Theorem. If f is differentiable at a, then f is continuous at a

When is the function not differentiable at $x = a$?

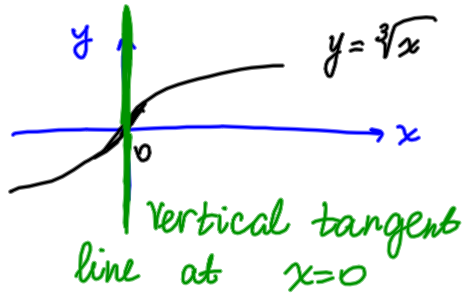
- f has a "corner" or "kink" at a



$f'(a)$ DNE

- f is discontinuous at a

- the curve $y = f(x)$ has a vertical tangent line at $x = a$



$f'(0)$ DNE

Besides being able to find the function $f(x)$ from the function $f(x)$, you can also determin the graph of $f(x)$ from the graph of $f(x)$.
Things to look for when sketching $f(x)$ from the given graph of $f(x)$:

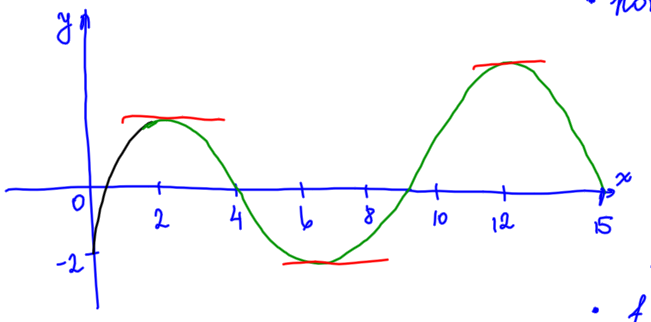
- Values of slopes of tangent lines to $f(x)$
- Points where $f(x)$ has horizontal tangents (slope of the tangent line $= 0$)
- Intervals where $f(x)$ is increasing or decreasing

$f'(x) > 0$ whenever $f(x)$ is increasing

$f'(x) < 0$ whenever $f(x)$ is decreasing

- Places where $f(x)$ is leveling off

Example 4. Given the graph of $f(x)$ below, sketch the graph of $f'(x)$.



• horizontal tangents at $x=2$, $x=6$ and $x=12$

$f'(2) = f'(6) = f'(12) = 0$

• f increases on $(0,2) \cup (6,12)$

$f'(x) > 0$ on $(0,2) \cup (6,12)$

• f decreases on $(2,6) \cup (12,15)$

$f'(x) < 0$ on $(2,6) \cup (12,15)$

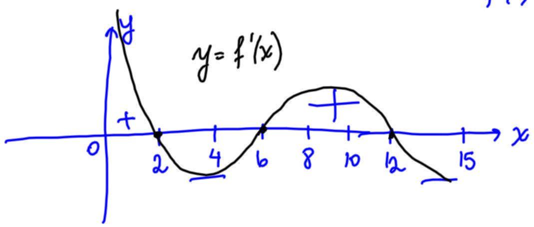


Table of derivatives.

1. $(c)' = 0$, c is a constant
2. $(x)' = 1$
3. $(x^2)' = 2x$
4. $(x^3)' = 3x^2$
5. $(x^n)' = n x^{n-1}$
6. $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$
7. $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$

$$(a f(x) \pm b g(x))' = a f'(x) \pm b g'(x)$$

Examples. $\left(x^2 + 4x + 2 - \frac{1}{x}\right)' = (x^2)' + (4x)' + (2)' - \left(\frac{1}{x}\right)'$

$$= \boxed{2x + 4 + \frac{1}{x^2}}$$

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$