4.1 Derivatives of Powers, Exponents, and Sums

Derivative notation for $y = f(x)$: $\quad f'(x) = y' = \dfrac{df}{dx} = \dfrac{dy}{dx} = \dfrac{d}{dx}[f(x)]$

1. $(C)' = 0$, C is a constant.
2. $(x)' = 1$,
3. $(x^2)' = 2x$,
4. $(x^3)' = 3x^2$
5. $(\sqrt{x})' = \dfrac{1}{2}x^{-1/2}$
6. $\left(\dfrac{1}{x}\right)' = -x^{-2}$
7. $(x^n)' = nx^{n-1}$
8. $(e^x)' = e^x$
9. $(\ln x)' = \dfrac{1}{x}$

Differentiation formulas

Suppose c is a constant and both functions $f(x)$ and $g(x)$ are differentiable, then

1. $(cf(x))' = cf'(x)$,
2. $(f(x) + g(x))' = f'(x) + g'(x)$,
3. $(f(x) - g(x))' = f'(x) - g'(x)$.

Example 1. Differentiate each function.

1. $f(x) = e + \pi$

   $f'(x) = 0$

2. $f(x) = x^7$

   $f'(x) = 7x^6$

3. $f(x) = x^{3.1}$

   $f'(x) = 3.1\, x^{2.1}$

4. $f(x) = 0.24\sqrt[3]{x}$ $= 0.24x^{1/3}$

$f'(x) = 0.24 \cdot \frac{1}{3} x^{\frac{1}{3}-1} = 0.08x^{-\frac{2}{3}}$

5. $f(x) = 2e^x - 1$

$f'(x) = 2e^x$

5'. $f(x) = x^e + e^x$

$f'(x) = e x^{e-1} + e^x$

6. $f(x) = \pi x + 2\ln x$

$f'(x) = \pi + \frac{2}{x}$

7. $f(x) = x^5 - 4x^3 + 2x - 3$

$f'(x) = 5x^4 - 12x^2 + 2$

8. $f(x) = 3x^{2/3} - 2x^{5/2} + x^{-3}$

$f'(x) = 3 \cdot \frac{2}{3} x^{\frac{2}{3}-1} - 2\frac{5}{2} x^{\frac{5}{2}-1} + (-3)x^{-3-1}$

$= 2x^{-1/3} - 5x^{3/2} - 3x^{-4}$

Example 2. Find the equation ~~of~~ of the tangent line to the curve $y = x + \sqrt{x}$ at the point $(\overset{a}{1}, \overset{f(a)}{2})$

$y - f(a) = f'(a)(x-a)$ – equation of the tangent line to $y = f(x)$ at the point where $x = a$

$f(x) = x + \sqrt{x}$, $a = 1$, $f(1) = 2$

$f'(x) = 1 + \frac{1}{2}x^{-1/2}$

$f'(1) = 1 + \frac{1}{2} = \frac{3}{2}$ – slope of the tangent line

$y - 2 = \frac{3}{2}(x-1)$ or $y = \frac{3}{2}x - \frac{3}{2} + 2$

$\boxed{y = \frac{3}{2}x + \frac{1}{2}}$

Rates of Change in Business and Economics

Marginal analysis is the study of the amount of change in the **dependent variable** that results from a single unit change in the **independent variable**.

If x represents the number of units of a product produced in some time interval, then

Total Cost $= C(x)$ and **Marginal Cost** $= C'(x)$

Total Revenue $= R(x)$ and **Marginal Revenue** $= R'(x)$

Total Profit ~~$= P(x) = R(x)C(x)$~~ and **Marginal Profit** $= P'(x)$

$P(x) = R(x) - C(x)$

Note. $C(x)$ represents the total cost of producing x items. **Exact cost of producing the** $(x+1)$**st item** $= C(x+1) - C(x)$.

Marginal cost and exact cost. If $C(x)$ is the total cost producing x items, the **marginal cost function approximates the exact cost of producing the** $(x+1)$**st item**

$C'(x)$

$$C'(x) \approx C(x+1) - C(x)$$

Similar statement can be made for $R(x)$ and $P(x)$.

Example 3. The total cost (in dollars) of producing x electric guitars is $C(x) = 1000 + 100x - 0.25x^2$

(a) Find the exact cost of producing the 51st guitar. $= C(51) - C(50)$

$C(51) =$ total cost of producing 51 guitars

$C(50) =$ total cost of producing 50 guitars.

Exact cost of producing the 51st guitar $= C(51) - C(50)$

$= 1000 + 100(51) - 0.25(51)^2 - 1000 - 100(50) + 0.25(50)^2 = \boxed{\$74.75}$

(b) Use the marginal cost to approximate the cost of producing the 51st guitar.

$C'(x) = 100 - 0.25(2)x$

$= 100 - 0.5x$

$C'(50)$ will approximate the cost of producing the 51st guitar.

$C'(50) = 100 - 0.5(50) = \$75.$

$C'(50) \approx C(51) - C(50)$

Example 4. The price-demand equation and the cost function for the production of table saws are given, respectively, by

$$x = 600 - 3p \text{ and } C(x) = 7200 + 6x$$

where x is a number of saws that can be sold at a price of \$p per saw and $C(x)$ is the total cost (in dollars) of producing x saws.

(a) Express p as a function of x.

$x = 600 - 3p$

$3p = 600 - x$

$p = 200 - \frac{x}{3}$

(b) Find the marginal cost. $C(x) = 7200 + 6x$

$C'(x) = \$6$

approximately, we need \$6 to produce a new saw.

(c) Find the revenue function.

$R(x) = xp(x) = 200x - \frac{x^2}{3}$

(d) Find the marginal revenue.

$R'(x) = 200 - \frac{2x}{3}$

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(e) Find $R'(150)$ and interpret this quantity.

$R'(150) = 200 - \frac{2(150)}{3} = 100$

approximation of revenue of producing 151st saw

(f) Find the profit function in terms of x.

$P(x) = R(x) - C(x)$

$= 200x - \frac{x^2}{3} - 7200 - 6x$

$= -\frac{x^2}{3} + 194x - 7200$

$$P(x) = -\frac{x^2}{3} + 194x - 7200$$

(g) Find the marginal profit.

$$P'(x) = -\frac{2x}{3} + 194$$

(h) Find $P'(150)$ and interpret this quantity.

$$P'(150) = -\frac{2}{3}(150) + 194$$

$$= 94$$

Approximation of profit of producing 151st saw.

4.2. Product and Quotient Rules.

- Product Rule: $\boxed{(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)}$
- Quotient Rule: $\boxed{\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}}$

Example. Find the derivative.

(a) $f(x) = (x^2 + 2x - 3\ln x)(e^x + \sqrt{x})$

$$f'(x) = (x^2+2x-3\ln x)'(e^x+\sqrt{x}) + (x^2+2x-3\ln x)(e^x+\sqrt{x})'$$

$$= \boxed{\left(2x+2-\frac{3}{x}\right)(e^x+\sqrt{x}) + (x^2+2x-3\ln x)\left(e^x+\frac{1}{2}x^{-1/2}\right)}$$