4.2. Derivatives of products and quotients.

Product rule. $\boxed{(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).}$

Example 1. Find $f'(x)$ for

1. $f(x) = x^3(x^2 - x + 5)$

2. $f(x) = (x^3 - 2x + 1)(x^2 + 1)$

$$f'(x) = \underbrace{(x^3 - 2x + 1)'}_{3x^2-2}(x^2+1) + (x^3 - 2x + 1)\underbrace{(x^2+1)'}_{2x}$$

$$= \boxed{(3x^2-2)(x^2+1) + (x^3-2x+1)(2x)}$$

3. $f(x) = (2x^2 + 1)e^x$.

$$f'(x) = (2x^2+1)'e^x + (2x^2+1)(e^x)'$$

$$= \boxed{4xe^x + (2x^2+1)e^x}$$

Quotient rule. $\boxed{\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}.}$

Example 2. Find f'

1. $f(x) = \dfrac{x^{2/3}+1}{x^3+1}$

$$f'(x) = \frac{(x^{2/3}+1)'(x^3+1) - (x^3+1)'(x^{2/3}+1)}{(x^3+1)^2}$$

$$= \frac{\frac{2}{3}x^{\frac{2}{3}-1}(x^3+1) - 3x^2(x^{2/3}+1)}{(x^3+1)^2}$$

$$= \boxed{\frac{\frac{2}{3}x^{-1/3}(x^3+1) - 3x^2(x^{2/3}+1)}{(x^3+1)^2}}$$

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2. $f(x) = \dfrac{\ln x}{x^3+1}$

$$f'(x) = \frac{(\ln x)'(x^3+1) - (x^3-1)'\ln x}{(x^3+1)^2} = \frac{\frac{1}{x}(x^3+1) - 3x^2\ln x}{(x^3+1)^2}$$

$$= \frac{\frac{x^3+1-3x^3\ln x}{x}}{(x^3+1)^2} = \boxed{\frac{x^3+1-3x^3\ln x}{x(x^3+1)^2}}$$

3. $f(x) = \dfrac{\sqrt[3]{x}}{e^x+x} = \dfrac{x^{1/3}}{e^x+x}$

$$f'(x) = \frac{(x^{1/3})'(e^x+x) - x^{1/3}(e^x+x)'}{(e^x+x)^2} = \frac{\frac{1}{3}x^{1/3-1}(e^x+x) - x^{1/3}(e^x+1)}{(e^x+x)^2}$$

$$= \boxed{\frac{x^{-2/3}(e^x+x) - 3x^{1/3}(e^x+1)}{3(e^x+x)^2}}$$

Example 3. Find $h'(x)$ for the functions below, where $f(x)$ is an unspecified differentiable function.

1. $h(x) = f(x)\ln(x)$

$$h'(x) = f'(x)\ln x + f(x)(\ln x)'$$
$$= \boxed{f'(x)\ln x + \frac{1}{x} f(x)}$$

2. $h(x) = \dfrac{xe^x + x^2}{f(x)}$

product rule

$$h'(x) = \frac{(xe^x + x^2)' f(x) - f'(x)(xe^x + x^2)}{[f(x)]^2}$$
$$= \frac{((x)'e^x + x(e^x)' + 2x) f(x) - f'(x)(xe^x + x^2)}{[f(x)]^2}$$
$$= \boxed{\frac{(e^x + xe^x + 2x) f(x) - f'(x)(xe^x + x^2)}{[f(x)]^2}}$$

Example 4. Find the value(s) of x where $f(x) = x^3(6x^2 - 5)$ has a horizontal tangent line.

horizontal tangent line = slope is zero = $f'(x)=0$

Find the values for x such that $f'(x)=0$.

$$f'(x) = \underbrace{3x^2}_{(x^3)'}(6x^2-5) + x^3\underbrace{(12x)}_{(6x^2-5)'}$$

$$= 3x^2(6x^2-5) + 12x^4$$

$$\frac{12x^4}{3x^2} = 4x^2$$

$$= 3x^2(6x^2-5+4x^2)$$

$$= \frac{3x^2(10x^2-5)}{3} = \frac{0}{3}$$

$$x^2(10x^2-5)=0$$

$$x^2=0 \quad \text{or} \quad 10x^2-5=0$$

$$\boxed{x=0}$$

$$\frac{10x^2}{10} = \frac{5}{10}$$

$$x^2 = \frac{1}{2}$$

$$\boxed{x = \frac{1}{\sqrt{2}} \quad \Big| \quad x = -\frac{1}{\sqrt{2}}}$$

If $C(x)$ is the cost of producing x items, then the **average cost**, denoted by $\bar{C}(x)$, is the cost of all items divided by the number of items, that is,

$$\boxed{\bar{C}(x) = \frac{C(x)}{x}} \implies C(x) = \bar{C}(x)\,x$$

The **marginal average cost** is given by $\boxed{\frac{d}{dx}[\bar{C}(x)]}$

Example 6. Find the formula for the marginal average cost in terms of the cost and marginal cost.

$$\frac{d}{dx}\left(\bar{C}(x)\right) = \left(\frac{C(x)}{x}\right)' = \frac{C'(x)\,x - (x)'C(x)}{x^2}$$

$$= \frac{C'(x)\,x - C(x)}{x^2} = \frac{C'(x)\,\cancel{x} - \cancel{x}\,\bar{C}(x)}{x^{\cancel{2}}}$$

$$= \boxed{\frac{C'(x) - \bar{C}(x)}{x}}$$

The marginal average cost is found by taking marginal cost less average cost and dividing the result by the number of items sold.

Example 7. If the cost function is given by $C(x) = x^3 - 2x^2 + 1$, find $\boxed{\left(\frac{T}{B}\right)' = \frac{T'B - B'T}{B^2}}$

1. Marginal cost $\boxed{C'(x) = 3x^2 - 4x}$

2. Average cost $\boxed{\overline{C}(x) = \frac{C(x)}{x} = \frac{x^3 - 2x^2 + 1}{x}}$

3. Marginal average cost $\left(\overline{C}(x)\right)' = \left(\frac{x^3 - 2x^2 + 1}{x}\right)'$

quotient rule $= \frac{(x^3 - 2x^2 + 1)'x - (x)'(x^3 - 2x^2 + 1)}{x^2}$

$$= \boxed{\frac{(3x^2 - 4x)x - (x^3 - 2x^2 + 1)}{x^2}}$$