Section 4.3. **The chain rule.**

A function m is a composite of functions f and g if

$$\boxed{m(x) = f(g(x))}$$

f is the outer function
g is the inner function

The domain of m is the set of all numbers x such that x is in the domain of g and $g(x)$ is in the domain of f.

Example 1. Let $f(x) = x^2$ and $g(x) = e^x$. Find $f(g(x))$ and $g(f(x))$.

$f(x)=x^2,\ g(x)=e^x$

$\boxed{f(g(x)) = (e^x)^2 = e^{2x}}$

$g(x)=e^x,\ f(x)=x^2$

$\boxed{g(f(x)) = e^{x^2}}$

Example 2. Write each function as a composition of two simpler functions.

1. $f(x) = \ln(x^2 + 3)$ $= g(h(x))$

   $\boxed{\begin{array}{l} g(x) = \ln x \\ h(x) = x^2+3 \end{array}}$

2. $f(x) = \sqrt[3]{3 - 4x^2}$ $= g(h(x))$

   $\boxed{\begin{array}{l} g(x) = \sqrt[3]{x} \\ h(x) = 3-4x^2 \end{array}}$

3. $f(x) = (x^6 + 3x^2 - 1)^5$ $= g(h(x))$

   $\boxed{\begin{array}{l} g(x) = x^5 \\ h(x) = x^6+3x^2-1 \end{array}}$

General power rule. If $u(x)$ is a differentiable function, n is any real number, then

$$\boxed{([u(x)]^n)' = n[u(x)]^{n-1}u'(x).}$$

Example 3. Find f' if

1. $f(x) = (2x+3)^5$

   $f'(x) = 5(2x+3)^{5-1}(2x+3)'$

   $= \boxed{5(2x+3)^4(2)}$

2. $f(x) = \sqrt{2-3x^3} = (2-3x^3)^{1/2}$

   $f'(x) = \frac{1}{2}(2-3x^3)^{1/2-1}(2-3x^3)'$

   $= \boxed{\frac{1}{2}(2-3x^3)^{-1/2}(-9x^2)}$

3. $f(x) = (\ln x + 2)^2$.

   $f'(x) = 2(\ln x+2)^{2-1}(\ln x+2)'$

   $= \boxed{2(\ln x+2)\frac{1}{x}}$

Chain rule. If $m(x) = f(g(x))$, then

$$\boxed{m'(x) = f'(g(x))g'(x).}$$

$$\left(e^{x\ln x}\right)' = e^{x\ln x}(x\ln x)' = e^{x\ln x}\left(\ln x + \frac{x}{x}\right)$$

$$\boxed{\left(e^{u(x)}\right)' = e^{u(x)}u'(x)}$$

Example 4. Let $f(x) = e^x$ and $g(x) = x^3 + 2x - 1$. Find

1. $[f(g(x))]'$

$$f(g(x)) = e^{x^3+2x-1}$$

$$= e^{x^3+2x-1}(x^3+2x-1)'$$

$$= \boxed{(3x^2+2)e^{x^3+2x-1}}$$

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$$f(x) = e^x, \quad g(x) = x^3+2x-1$$

2. $[g(f(x))]'$.

$$g(f(x)) = (e^x)^3 + 2(e^x) - 1$$

$$= e^{3x} + 2e^x - 1$$

$$[g(f(x))]' = (e^{3x} + 2e^x - 1)' = (e^{3x})' + (2e^x) - (1)'$$

$$= e^{3x}(3x)' + 2e^x$$

$$= \boxed{3e^{3x} + 2e^x}$$

Example 5. Find the derivative of the function.

1. $f(x) = (1 - x^3)^4 \ln x$

Product Rule:

$$f'(x) = \left((1-x^3)^4\right)' \ln x + (1-x^3)^4 (\ln x)'$$
$$= 4(1-x^3)^3(1-x^3)' \ln x + (1-x^3)^4 \frac{1}{x}$$
$$= \boxed{4(1-x^3)^3(-3x^2)\ln x + \frac{(1-x^3)^4}{x}}$$

2. $f(x) = (7x + 3)^3(x^2 - 4)^6$

Product Rule.

$$f'(x) = \left((7x+3)^3\right)'(x^2-4)^6 + (7x+3)^3\left((x^2-4)^6\right)'$$
$$= 3(7x+3)^2\underbrace{(7x+3)'}_{7}(x^2-4)^6 + (7x+3)^3\, 6(x^2-4)^5\underbrace{(x^2-4)'}_{2x}$$
$$= \boxed{3(7x+3)^2(7)(x^2-4)^6 + (7x+3)^3 6(x^2-4)^5(2x)}$$

3. $f(x) = e^{2x}\sqrt{\ln x} = e^{2x} \cdot (\ln x)^{1/2}$

Product Rule:

General Power Rule

$$f'(x) = (e^{2x})'(\ln x)^{1/2} + e^{2x}\left((\ln x)^{1/2}\right)'$$
$$= e^{2x}\underbrace{(2x)'}_{2}(\ln x)^{1/2} + e^{2x}\cdot\frac{1}{2}(\ln x)^{1/2-1}\underbrace{(\ln x)'}_{\frac{1}{x}}$$
$$= \boxed{2e^{2x}(\ln x)^{1/2} + \frac{e^{2x}}{2x}(\ln x)^{-1/2}}$$

4. $f(x) = \sqrt[3]{\sqrt{x} + 1} = (x^{1/2}+1)^{1/3}$

General Power Rule:

$$f'(x) = \frac{1}{3}(x^{1/2}+1)^{1/3-1}(x^{1/2}+1)'$$
$$= \boxed{\frac{1}{3}(x^{1/2}+1)^{-\frac{2}{3}}\left(\frac{1}{2}x^{-1/2}\right)}$$