Section 4.4 **Derivatives of exponential and logarithmic functions.**

- $\boxed{(e^x)' = e^x}$
- $\boxed{(\ln x)' = \frac{1}{x}}$
- $\boxed{(a^x)' = a^x \ln a,}$ where $a > 0$
- $\boxed{(\log_b x)' = \frac{1}{x \ln b}}$, where $b > 0$

Example 1. Differentiate the following function.

1. $f(x) = \sqrt[3]{x}e^x = x^{1/3}e^x$

Product Rule: $f'(x) = (x^{1/3})'e^x + x^{1/3}(e^x)'$

$= \frac{1}{3}x^{\frac{1}{3}-1}e^x + x^{1/3}e^x$

$= \boxed{\frac{1}{3}x^{-\frac{2}{3}}e^x + x^{1/3}e^x}$

2. $f(x) = \sqrt{2^x + 1} = (2^x+1)^{1/2}$

General Power Rule: $f'(x) = \frac{1}{2}(2^x+1)^{1/2-1}(2^x+1)'$

$(2^x)' = 2^x \ln 2$

$= \boxed{\frac{1}{2}(2^x+1)^{-1/2}(2^x)(\ln 2)}$

3. $f(x) = \dfrac{x}{1 - 3^x}$

Quotient Rule: $f'(x) = \dfrac{(x)'(1-3^x) - x(1-3^x)'}{(1-3^x)^2}$

$1 - 3^x = 1 + (-1)(3^x)$

$= \dfrac{1-3^x - x(-3^x \ln 3)}{(1-3^x)^2}$

$= \boxed{\dfrac{1-3^x + x(3^x)(\ln 3)}{(1-3^x)^2}}$

$\log x = \log_{10} x$ $\quad (\log x)' = \dfrac{1}{x \ln 10}$

4. $f(x) = \dfrac{\log(x^2)}{x + 1} = \dfrac{2\log x}{x+1}$

Quotient Rule: $f'(x) = 2\dfrac{(\log x)'(x+1) - (\log x)(x+1)'}{(x+1)^2} = 2\dfrac{\frac{x+1}{x \ln 10} - \log x}{(x+1)^2} = 2\dfrac{\frac{x+1-(\log x)(x)(\ln 10)}{x \ln 10}}{(x+1)^2}$

$= \boxed{2\dfrac{x+1-(\log x)(x)(\ln 10)}{x(\ln 10)(x+1)^2}}$

5. $f(x) = x^2 \log_2(\sqrt{x}) = x^2 \log_2(x^{1/2}) = \frac{1}{2}x^2 \log_2 x$

Product Rule: $f'(x) = \frac{1}{2}(x^2)'(\log_2 x) + \frac{1}{2}(x^2)(\log_2 x)'$

$= \boxed{\frac{1}{2}(2x)(\log_2 x) + \frac{1}{2}(x^2)\frac{1}{x \ln 2}}$

6. $f(x) = \sqrt{\ln x} = (\ln x)^{1/2}$

General Power Rule: $f'(x) = \frac{1}{2}(\ln x)^{1/2 - 1}(\ln x)'$

$= \boxed{\frac{1}{2}(\ln x)^{-1/2}\left(\frac{1}{x}\right)}$

Chain Rule for exponential and logarithmic functions:

- $\boxed{(e^{f(x)})' = f'(x)e^{f(x)}}$
- $\boxed{(\ln[f(x)])' = \dfrac{f'(x)}{f(x)}}$
- $\boxed{(a^{f(x)})' = f'(x)a^{f(x)}\ln a}$, where $a > 0$
- $\boxed{\left(\log_b[f(x)]\right)' = \dfrac{f'(x)}{f(x)\ln b}}$, where $b > 0$

Example 2. Find the derivative.

1. $f(x) = e^{x^2+x+1}$

$f'(x) = (x^2+x+1)'e^{x^2+x+1}$

$= \boxed{(2x+1)e^{x^2+x+1}}$

2. $f(x) = 2^{4x^6-5x^2+2}$

$f'(x) = (4x^6-5x^2+2)'\left(2^{4x^6-5x^2+2}\right)(\ln 2)$

$= \boxed{(24x^5-10x)\left(2^{4x^6-5x^2+2}\right)(\ln 2)}$

3. $f(x) = 3\log_9(5x^2+3x-2)$

$f'(x) = 3\dfrac{(5x^2+3x-2)'}{(5x^2+3x-2)(\ln 9)}$

$= \boxed{3\dfrac{(10x+3)}{(5x^2+3x-2)(\ln 9)}}$

4. $f(x) = [\log(1+10^x)]^3$

General Power Rule: $f'(x) = 3\left(\log(1+10^x)\right)^2\left(\log(1+10^x)\right)'$

$= 3\left(\log(1+10^x)\right)^2 \dfrac{(1+10^x)'}{(1+10^x)\ln 10}$

$= 3\left(\log(1+10^x)\right)^2 \dfrac{10^x \cancel{\ln 10}}{(1+10^x)\cancel{\ln 10}}$

$= \boxed{3\left(\log(1+10^x)\right)^2 \dfrac{10^x}{1+10^x}}$

5. $f(x) = \ln[(2x+1)^5(5x-3)^7] = \ln(2x+1)^5 + \ln(5x-3)^7$

$= 5\ln(2x+1) + 7\ln(5x-3)$

$(\ln f(x))' = \frac{f'(x)}{f(x)}$

$f'(x) = 5\frac{(2x+1)'}{2x+1} + 7\frac{(5x-3)'}{5x-3}$

$= \boxed{\frac{10}{2x+1} + \frac{35}{5x-3}}$

6. $f(x) = \ln\left[\frac{x^6(3x+6)^9}{(4x-10)^{11}}\right] = \ln(x^6) + \ln(3x+6)^9 - \ln(4x-10)^{11}$

$= 6\ln x + 9\ln(3x+6) - 11\ln(4x-10)$

$f'(x) = \frac{6}{x} + 9\frac{1}{3x+6}(3x+6)' - 11\frac{1}{4x-10}(4x-10)'$

$= \boxed{\frac{6}{x} + 9\frac{3}{3x+6} - 11\frac{4}{4x-10}}$

7. $f(x) = \ln(xe^{x^4}) = \ln x + \ln(e^{x^4})$

$= \ln x + x^4 \underbrace{\ln e}_{1}$

$= \ln x + x^4$

$\boxed{f'(x) = \frac{1}{x} + 4x^3}$

Example 3. If \$2,400 is invested in a savings account offering interest at a rate of 3.5% per year, compounded continuously, how fast is the balance growing after 9 years? (Round your answer to the nearest cent.)

$A(t) = Pe^{rt}$

$P = 2400, \quad r = 3.5\% = 0.035$

$A(t) = 2400e^{0.035t}$

$A'(9) - ?$

$A'(t) = 2400e^{0.035t}(0.035t)'$

$= 2400(0.035)e^{0.035t}$

$A'(9) = 2400(0.035)e^{0.035(9)} \approx 115.10$

Example 4. Suppose the price and the demand of the commodity is related by $p(x) = e^{-2x}$. Find the marginal revenue $R'(x)$, and find where the marginal revenue is zero.

$R(x) = xp(x) = xe^{-2x}$

$R'(x) = (x)'e^{-2x} + x\underbrace{(e^{-2x})'}_{}$ chain Rule

$= e^{-2x} + xe^{-2x}(-2x)'$

$= e^{-2x} - 2xe^{-2x} = 0$

$e^{-2x}(1-2x) = 0$

e^{-2x} is never zero.

$1 - 2x = 0$

$\boxed{x = \frac{1}{2}}$