Section 4.5 **Elasticity of demand.**

Question: When will a price increase lead to an increase in revenue? To answer this question we will use the notion of *elasticity of demand*.

The **relative rate of change** of a function $f(x)$ is $\boxed{\dfrac{f'(x)}{f(x)}}$. The **percentage rate of change** is $\boxed{100 \times \dfrac{f'(x)}{f(x)}}$.

Since

$$(\ln[f(x)])' = \frac{f'(x)}{f(x)},$$

the relative rate of change of $f(x)$ is the derivative of the logarithm of $f(x)$. This is also referred to as the **logarithmic derivative** of $f(x)$.

Example 1. Find the relative rate of change of $f(x) = 50x - 0.01x^2$.

relative rate of change $= \dfrac{f'(x)}{f(x)} = \dfrac{(50x - 0.01x^2)'}{50x - 0.01x^2}$

$= \boxed{\dfrac{50 - 0.02x}{50x - 0.01x^2}}$

Logarithmic derivatives and relative rates are used by economists to study the relationship among price changes, demand, and revenue. For most products, demand is assumed to be a decreasing function of price. That is, price increases result in lower demand, and price decreases result in higher demand.

Economists use **elasticity of demand** to study the relationship between changes in price and changes in demand.

If price and demand are related by $x = f(p)$, then the elasticity of demand is given by

$$\boxed{E(p) = \frac{\text{relative rate of change of demand}}{\text{relative rate of change of price}} = -\frac{pf'(p)}{f(p)}.}$$

$\boxed{(\text{change of demand}) = E(p)(\text{change of price})}$

Example 2. Given the price-demand equation

$$p + 0.005x = 30. \leftarrow \text{solve for } x$$

1. Find the elasticity of demand $E(p) = \dfrac{-p f'(p)}{f(p)}$

$$p + 0.005x = 30$$
$$\frac{0.005x}{0.005} = \frac{30-p}{0.005}$$
$$x = \frac{30}{0.005} - \frac{p}{0.005}$$
$$x = \boxed{6000 - 200p = f(p)}$$
$$E(p) = -\frac{p(6000-200p)'}{(6000-200p)} = -\frac{p(-200)}{6000-200p} = \frac{200p}{6000-200p} = \boxed{\frac{p}{30-p}}$$

elasticity of demand

2. What is elasticity of demand if p=\$10? If this price is increased by 10%, what is the approximate change in demand?

$$E(10) = \frac{10}{30-10} = 0.5$$
$$(\text{change in demand}) = E(10)(\text{change in price})$$
$$= 0.5\,(10\%) = \boxed{5\%}$$

3. What is elasticity of demand if p=\$25? If this price is increased by 10%, what is the approximate change in demand?

$$E(25) = \frac{25}{30-25} = 5$$
$$(\text{change in demand}) = E(25)(10\%)$$
$$= 5(10\%) = \boxed{50\%}$$

4. What is elasticity of demand if p=\$15? If this price is increased by 10%, what is the approximate change in demand?

$$E(15) = \frac{15}{30-15} = 1$$
$$(\text{change in demand}) = E(15)(10\%)$$
$$= 10\%$$

- If $\boxed{0 < E(p) < 1}$ then the demand is inelastic, that is the demand is not sensitive to changes in price. A change in price produces a smaller change in demand.
- If $\boxed{E(p) > 1}$, then the demand is elastic, that is the demand is sensitive to changes in price. A change in price produces a larger change in demand.
- If $\boxed{E(p) = 1}$, then the demand is unit. A change in price produces the same change in demand.

Now we want to see how revenue and elasticity are related.

$$\text{Revenue} = (\text{demand }) \times \ (\text{price}) = xp = pf(p).$$

$$\boxed{R'(p) = f(p)(1 - E(p)).}$$

Since $f(p) > 0$, $R'(p)$ and $1 - E(p)$ always have the same sign.

If $E(p) < 1$ (demand is inelastic), then $1 - E(p) > 0$ and $R'(p) > 0$.

If $E(p) > 1$ (demand is elastic), then $1 - E(p) < 0$ and $R'(p) < 0$.

Revenue and elasticity of demand.

- Demand is inelastic:
  1. A price increase will increase revenue.
  2. A price decrease will decrease revenue.
- Demand is elastic:
  1. A price increase will decrease revenue.
  2. A price decrease will increase revenue.

Example 3. The price-demand equation for hamburgers at a fast-food restaurant is

$$x + 400p = 2000. \Rightarrow \boxed{x = 2000 - 400p = f(p)}$$

Currently, the price of a hamburger is \$2. If the price is increased by 10%, will revenue increase or decrease?

$$E(p) = -\frac{pf'(p)}{f(p)} = \frac{-p(2000-400p)'}{2000-400p} = \frac{-p(-400)}{2000-400p}$$

$$= \frac{400p}{2000-400p} = \frac{p}{5-p}$$

$$E(2) = \frac{2}{5-2} = \frac{2}{3} < 1 \quad \text{inelastic}$$

$\boxed{\text{the revenue will increase.}}$