Section 5.2. **Second derivative.**

For $y = f(x)$, the **second derivative** of f, provided that it exists, is

$$\boxed{f''(x) = \frac{d}{dx} f'(x)} = y'' = \frac{d^2y}{dx^2}$$

Other notation for $f''(x)$ are

$$\frac{d^2y}{dx^2} \quad y''$$

Example 1. Find f'' for

1. $f(x) = -6x^{-2} + 12x^{-3}$

$$f'(x) = -6(-2)x^{-2-1} + 12(-3)x^{-3-1}$$
$$= 12x^{-3} - 36x^{-4}$$
$$f''(x) = (12x^{-3} - 36x^{-4})'$$
$$= 12(-3)x^{-3-1} - 36(-4)x^{-4-1}$$
$$= \boxed{-36x^{-4} + 144x^{-5}}$$

2. $f(x) = xe^x$

$$f'(x) \overset{\text{Product Rule}}{=} (x)'e^x + x(e^x)'$$
$$= e^x + xe^x$$
$$f''(x) = (e^x + xe^x)'$$
$$= (e^x)' + (xe^x)'$$
$$= e^x + \underbrace{e^x + xe^x}$$
$$= \boxed{2e^x + xe^x}$$

3. $f(x) = 2\sqrt[3]{x^2 - 4} = 2(x^2-4)^{1/3}$

$$f'(x) \overset{\text{General Power Rule}}{=} 2\frac{1}{3}(x^2-4)^{1/3-1}(x^2-4)'$$
$$= \frac{2}{3}(x^2-4)^{-\frac{2}{3}}(2x)$$
$$= \frac{4}{3}x(x^2-4)^{-\frac{2}{3}}$$
$$f''(x) = (f'(x))'$$
$$\overset{\text{Product Rule}}{=} \frac{4}{3}\left\{(x)'(x^2-4)^{-\frac{2}{3}} + x\overbrace{\left[(x^2-4)^{-\frac{2}{3}}\right]'}^{\text{General Power Rule}}\right\}$$
$$= \frac{4}{3}\left\{(x^2-4)^{-\frac{2}{3}} + x\left(-\frac{2}{3}\right)(x^2-4)^{-\frac{2}{3}-1}(x^2-4)'\right\}$$
$$= \boxed{\frac{4}{3}\left\{(x^2-4)^{-\frac{2}{3}} + x\left(-\frac{2}{3}\right)(x^2-4)^{-\frac{5}{3}}(2x)\right\}}$$

Higher derivatives: $f^{(n)}(x) = \left(f^{(n-1)}(x)\right)' = \dfrac{d}{dx}\left(f^{(n-1)}(x)\right)$, $n \geq 1$.

Example 2. Find the third derivative of the function $f(x) = \dfrac{x^2}{x+1}$.

Quotient Rule

$$f'(x) = \frac{(x^2)'(x+1) - x^2(x+1)'}{(x+1)^2} = \frac{2x(x+1) - x^2}{(x+1)^2} = \frac{2x^2 + 2x - x^2}{(x+1)^2} = \frac{x^2+2x}{(x+1)^2}$$

$$f''(x) = \frac{(x^2+2x)'(x+1)^2 - (x^2+2x)\left[(x+1)^2\right]'}{\left[(x+1)^2\right]^2}$$ General Power Rule

$$= \frac{(2x+2)(x+1)^2 - (x^2+2x)2(x+1)^{2-1}(x+1)'}{(x+1)^4}$$

$$= \frac{(2x+2)(x+1)^2 - 2(x^2+2x)(x+1)}{(x+1)^4}$$

$$= \frac{\cancel{(x+1)}\left[(2x+2)(x+1) - 2(x^2+2x)\right]}{(x+1)^{\cancel{4}3}} = \frac{\cancel{2x^2} + \cancel{2x} + \cancel{2x} + 2 - \cancel{2x^2} - \cancel{4x}}{(x+1)^3}$$

$$= \frac{2}{(x+1)^3} = 2(x+1)^{-3}$$

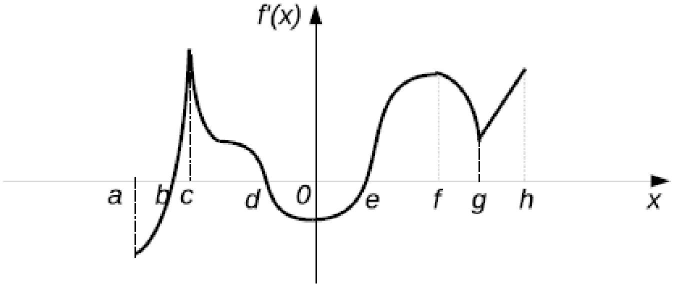
$$f'''(x) = \left(2(x+1)^{-3}\right)' = 2(-3)(x+1)^{-3-1}(x+1)'$$

$$= \boxed{-6(x+1)^{-4}}$$

The graph of the function f is concave upward (CU) on the interval (a, b) if $f'(x)$ is increasing on (a, b) and is concave downward (CD) on the interval (a, b) if $f'(x)$ is decreasing on (a, b)

An inflection point is a point on the graph of the function where the concavity changes.

Example 3. Given the graph of $f'(x)$ (the derivative)



1. Identify intervals on which f is concave upward. ($f'(x)$ is increasing)

   $\boxed{(a, c) \cup (0, f) \cup (g, h)}$

2. Identify intervals on which f is concave downward. ($f'(x)$ decreases)

   $\boxed{(c, 0) \cup (f, g)}$

3. Find the x-coordinates of inflection points. $\boxed{x = c}$ $\boxed{x = 0}$ $\boxed{x = f}$ $\boxed{x = g}$

Concavity test.

- If $f''(x) > 0$ on an interval, then f is CU on this interval.
- If $f''(x) < 0$ on an interval, then f is CD on this interval.

The second derivative test. Suppose f'' is continuous near c.

1. If $f'(c) = 0$ and $f''(c) > 0$, then f has a **local min** at c. $x=c$ concave upward
2. If $f'(c) = 0$ and $f''(c) < 0$, then f has a **local max** at c. $x=c$ concave downward

If $y = f(x)$ is continuous on (a, b) and has an inflection point at $x = c$, then either $\boxed{f''(c) = 0}$ or $\boxed{f''(c) \text{ does not exist}}$.

Example 4. Find inflection points and the intervals of which the graph of

$$f(x) = x^4 - 2x^3 - 36x + 12$$

is CU/CD.

Step 1. Find inflection points:

$f'(x) = 4x^3 - 6x^2 - 36$

$f''(x) = 12x^2 - 12x = 0$

$12x(x-1) = 0$

$\boxed{x_1 = 0, \ x_2 = 1 \ - \text{ inflection points}}$

Step 2. Construct the sign chart for $f''(x)$.

f: $\frown$ $\smile$ $\frown$

f'': $+$ $-$ $+$ x

0 1

$f''(-1) = 12(-1)^2 - 12(-1) = 24 > 0$

$f''\left(\frac{1}{2}\right) = 12\left(\frac{1}{2}\right)^2 - 12\left(\frac{1}{2}\right) = 3 - 6 = -3 < 0$

$f''(2) = 12(2)^2 - 12(2) = 24 > 0$

$\boxed{\begin{array}{l} f \text{ is CU on } (-\infty, 0) \cup (1, \infty) \\ f \text{ is CD on } (0, 1) \end{array}}$