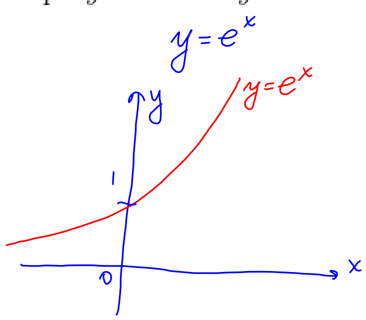
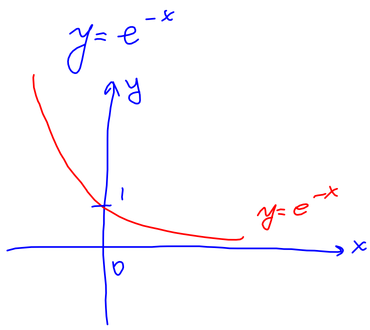
Section 5.3. **Limits at infinity.**

Graph $y = e^x$ and $y = e^{-x}$ below:




Find:

$\lim_{x\to\infty} e^x = \infty$

$\lim_{x\to\infty} e^{-x} = 0$

$\lim_{x\to-\infty} e^x = 0$

$\lim_{x\to-\infty} e^{-x} = \infty$

**Example 1.** Find the limit:

1. $\lim_{x\to\infty} \frac{4}{6+e^{3x}} = \frac{4}{6+\lim_{x\to\infty} e^{3x}} = \frac{4}{6+\text{"}\infty\text{"}} = \frac{4}{\text{"}\infty\text{"}} = \boxed{0}$ $\boxed{\frac{\text{number}}{\text{infinity}} = 0}$

2. $\lim_{x\to-\infty} \frac{4}{6+e^{3x}} = \frac{4}{6+\lim_{x\to-\infty} e^{3x}} = \frac{4}{6+0} = \frac{4}{6} = \boxed{\frac{2}{3}}$

3. $\lim_{x\to\infty} \frac{2}{5+e^{-9x}} = \frac{2}{5+\lim_{x\to\infty} e^{-9x}} = \frac{2}{5+0} = \boxed{\frac{2}{5}}$

4. $\lim_{x\to-\infty} \frac{2}{5+e^{-9x}} = \frac{2}{5+\lim_{x\to-\infty} e^{-9x}} = \frac{2}{5+\text{"}\infty\text{"}} = \boxed{0}$

$$\lim_{x\to\infty} e^{-7x} = 0$$

5. $\displaystyle \lim_{x\to\infty} \frac{13+8e^{-7x}}{8+2e^{-7x}} = \frac{13+8\lim_{x\to\infty} e^{-7x}}{8+2\lim_{x\to\infty} e^{-7x}} = \frac{13+0}{8+0} = \boxed{\frac{13}{8}}$

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$$\lim_{x\to-\infty} e^{-7x} = \infty \qquad \frac{1}{e^{-7x}} = e^{7x}$$

6. $\displaystyle \lim_{x\to-\infty} \frac{13+8e^{-7x}}{8+2e^{-7x}} = \left|\frac{\infty}{\infty}\right| = \lim_{x\to-\infty} \frac{\cancel{e^{-7x}}\left(\frac{13}{e^{-7x}}+8\right)}{\cancel{e^{-7x}}\left(\frac{8}{e^{-7x}}+2\right)} = \lim_{x\to-\infty} \frac{13e^{7x}+8}{8e^{7x}+2}$ $\quad \lim_{x\to-\infty} e^{7x} = 0$

$$= \frac{13\lim_{x\to-\infty} e^{7x}+8}{8\lim_{x\to-\infty} e^{7x}+2} = \frac{0+8}{0+2} = \boxed{4}$$

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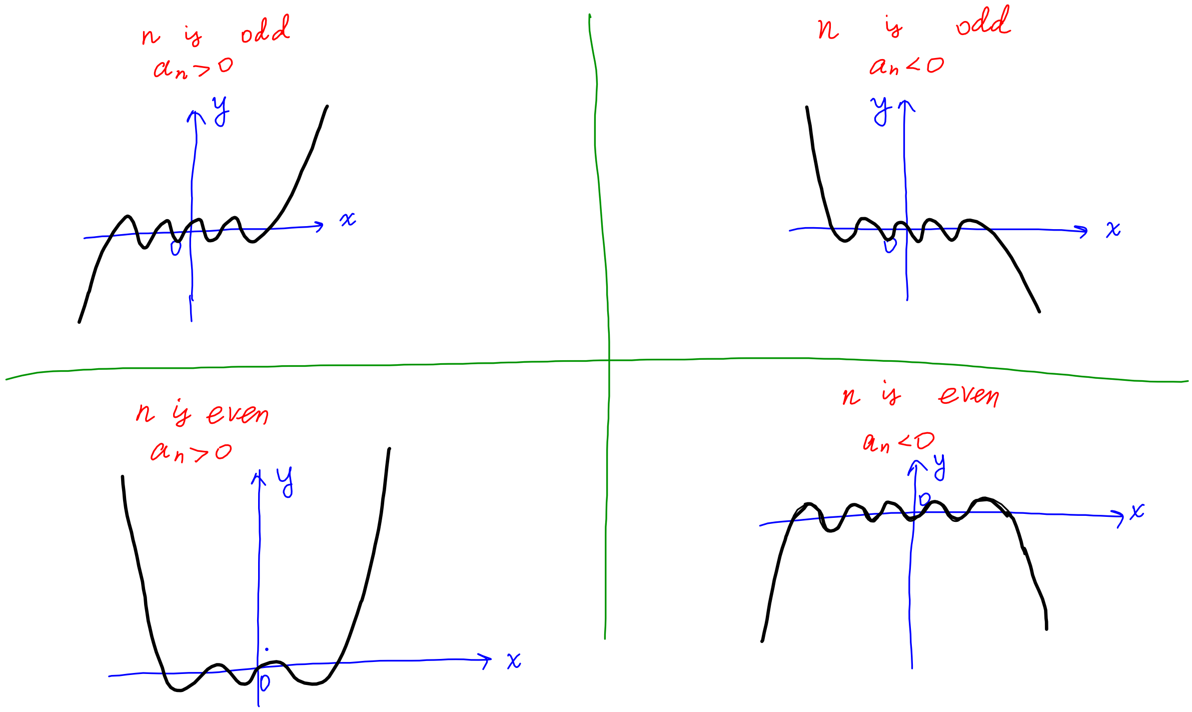
7. $\displaystyle \lim_{x\to\infty} \frac{21-17e^{8x}}{10+31e^{8x}} = \left|\frac{\infty}{\infty}\right|$ $\quad \lim_{x\to\infty} e^{8x} = \infty \qquad \frac{1}{e^{8x}} = e^{-8x}$

$$= \lim_{x\to\infty} \frac{\cancel{e^{8x}}\left(\frac{21}{e^{8x}}-17\right)}{\cancel{e^{8x}}\left(\frac{10}{e^{8x}}+31\right)} = \lim_{x\to\infty} \frac{21e^{-8x}-17}{10e^{-8x}+31} \qquad \lim_{x\to\infty} e^{-8x} = 0$$

$$= \frac{21\lim_{x\to\infty} e^{-8x}-17}{10\lim_{x\to\infty} e^{-8x}+31} = \boxed{-\frac{17}{31}}$$

**End behavior of polynomials:**

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$




**Example 2.** Find the limit:

1. $\lim\limits_{x\to\infty} (x^3 + x^4) = \infty$

2. $\lim\limits_{x\to-\infty} (x^3 + x^4) = \infty$

3. $\lim\limits_{x\to\infty} (x^3 - x^4) = -\infty$

4. $\lim\limits_{x\to-\infty} (x^3 - x^4) = -\infty$

5. $\lim\limits_{x\to\infty} (x^4 + x^5) = \infty$

6. $\lim\limits_{x\to-\infty} (x^4 + x^5) = -\infty$

7. $\lim\limits_{x\to\infty} (x^4 - x^5) = -\infty$

8. $\lim\limits_{x\to-\infty} (x^4 - x^5) = \infty$

A line $y = b$ is a horizontal asymptote of the graph of $y = f(x)$ if either $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$.

**End behavior of rational functions:**

$$\lim_{x\to\infty} \frac{a_n x^n + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0} = \lim_{x\to\infty} \frac{a_n x^n}{b_m x^m} = \begin{cases} \frac{a_n}{b_m}, & \text{if } n = m \\ 0, & \text{if } n < m \\ \infty, & \text{if } n > m, b_m > 0 \\ -\infty, & \text{if } n > m, b_m < 0 \end{cases}$$

$$\lim_{x\to-\infty} \frac{a_n x^n + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0} = \lim_{x\to\infty} \frac{a_n x^n}{b_m x^m} = \begin{cases} \frac{a_n}{b_m}, & \text{if } n = m \\ 0, & \text{if } n < m \\ -\infty, & \text{if } n > m, b_m > 0 \\ \infty, & \text{if } n > m, b_m < 0 \end{cases}$$

**A rational function can have at most one horizontal asymptote.**

**Example 3.** Find the limit:

1. $\lim_{x\to\infty} \frac{\boxed{x^3}+4x^2+3x-2}{x-\boxed{3x^3}} = \boxed{-\frac{1}{3}} = \lim_{x\to\infty} \frac{x^3}{-3x^3}$

2. $\lim_{x\to-\infty} \frac{x^3+4x^2+3x-2}{x-3x^3} = \lim_{x\to-\infty} \frac{\cancel{x^3}}{-3\cancel{x^3}} = \boxed{-\frac{1}{3}}$

3. $\lim_{x\to\infty} \frac{x^{10}-7}{9-5x^5} = \lim_{x\to\infty} \frac{x^{\cancel{10}\,5}}{-5\cancel{x^5}} = \lim_{x\to\infty} \frac{x^5}{-5} = \boxed{-\infty}$

4. $\lim_{x\to-\infty} \frac{x^{10}-7}{9-5x^5} = \lim_{x\to-\infty} \frac{x^{\cancel{10}\,5}}{-5\cancel{x^5}} = \lim_{x\to-\infty} \frac{x^5}{-5} = \boxed{\infty}$

5. $\lim_{x\to\infty} \frac{x^2+8}{2-3x^3} = \lim_{x\to\infty} \frac{\cancel{x^2}}{-3\cancel{x^3}} = \lim_{x\to\infty} \frac{1}{-3x} = \boxed{0}$

6. $\lim_{x\to-\infty} \frac{x^2+8}{2-3x^3} = \lim_{x\to-\infty} \frac{x^2+8}{2-3x^3} = \lim_{x\to-\infty} \frac{\cancel{x^2}}{-3\cancel{x^3}} = \lim_{x\to\infty} \frac{1}{-3x} = \boxed{0}$

**Example 4.** Find the horizontal asymptotes of the function:

1. $y = \dfrac{7x^2+3}{9x^3-80x-9}$

Equation: $y=b$, $b = \lim\limits_{x\to\infty} y(x)$ or $b = \lim\limits_{x\to-\infty} y(x)$

$$\lim_{x\to\infty} \frac{7x^2+3}{9x^3-80x-9} = \lim_{x\to\infty} \frac{7\cancel{x^2}}{9x^{\cancel{3}}} = \lim_{x\to\infty} \frac{7}{9x} = 0$$

The equation of the horizontal asymptote: $\boxed{y=0}$

2. $y = \dfrac{3x^2+7}{8x^2-71x-9}$

$$\lim_{x\to\infty} \frac{3x^2+7}{8x^2-71x-9} = \lim_{x\to\infty} \frac{3\cancel{x^2}}{8\cancel{x^2}} = \frac{3}{8}$$

Horizontal asymptote $\boxed{y = \frac{3}{8}}$

3. $y = \dfrac{11x^3+7}{5x^2-24x-5}$

$$\lim_{x\to\infty} \frac{11x^3+7}{5x^2-24x-5} = \lim_{x\to\infty} \frac{11x^{\cancel{3}}}{5\cancel{x^2}} = \lim_{x\to\infty} \frac{11x}{5} = \infty$$

$\boxed{\text{NO Horizontal asymptotes.}}$