Section 5.5 **Absolute maxima and minima.**

If $\boxed{f(c) \geq f(x)}$ for all x in the domain of f, then $f(c)$ is called the absolute maximum value of f.
If $\boxed{f(c) \leq f(x)}$ for all x in the domain of f, then $f(c)$ is called the absolute minimum value of f.
The absolute min and the absolute max are referred to as **absolute extrema.**

A function f is continuous on a closed interval $[a, b]$ has both an absolute maximum value and an absolute minimum value on that interval.

Locating absolute extrema. Absolute extrema (if they exist) must always occur at critical values or at endpoints.

Finding absolute extrema on a closed interval.

1. Check to make certain that f is continuous over $[a, b]$.
2. Find the critical values in the interval (a, b).
3. Evaluate f at the endpoints a and b and at the critical values found in Step 2.
4. The absolute maximum $f(x)$ on $[a, b]$ is the largest value found in Step 3.
5. The absolute minimum $f(x)$ on $[a, b]$ is the smallest value found in Step 3.

Example 1. Find the absolute maximum and minimum, if either exists, for the function $f(x) = x^3 - 6x^2 + 9x - 6$ on the interval

1. $[-1, 5]$

Step 1. Critical values for $f(x)$.

$$f'(x) = \frac{3x^2 - 12x + 9}{3} = \frac{0}{3}$$

$$x^2 - 4x + 3 = 0$$

$$(x-1)(x-3) = 0$$

$$x_1 = 1, \quad x_2 = 3.$$

Step 2.

$$f(1) = 1 - 6 + 9 - 6 = -2$$

$$f(3) = 27 - 6(9) + 9(3) - 6 = -6$$

$$f(-1) = -1 - 6 - 9 - 6 = \boxed{-22 \text{ abs. min}}$$

$$f(5) = 125 - 6(25) + 9(5) - 6 = \boxed{14 \text{ abs. max}}$$

2. $[-1, 3]$

$f'(x)=0$ @ $x=1,\ x=3.$

$f(-1) = \boxed{-22 \text{ abs. min}}$

$f(1) = \boxed{-2 \text{ abs. max}}$

$f(3) = -6$

3. $[2, 5]$

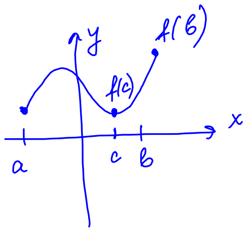
$f'(x)=0$ @ $x=1,\ x=3.$

not in the interval.

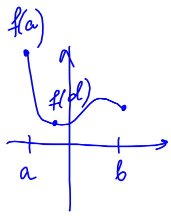
$f(3) = \boxed{-6 - \text{abs. min}}$

$f(2) = 8-6(4)+9(2)-6=-4$

$f(5) = \boxed{14 \text{ abs. max}}$



y
f(b)
f(c)
x
a
c
b



f(a)
y
f(d)
a
b

The second derivative test Suppose f'' is continuous near c.

1. If $f'(c) = 0$ and $f''(c) > 0$, then f has an absolute (local) min at c.
2. If $f'(c) = 0$ and $f''(c) < 0$, then f has an absolute (local) max at c.

Note. The second derivative test cannot be applied when $f''(c) = 0$ or $f''(c)$ does not exist.

Example 2. Assuming that the function $f(x)$ is continuous on the interval $(-\infty, \infty)$, indicate whether each of the points listed below is a local maximum, local minimum or neither. If this cannot be determined from the stated information, then indicate this in your answer.

1. $(2, f(2))$ if $f'(2) = 0$ and $f''(2) > 0$

   $(2, f(2))$ - local min

2. $(-1, f(-1))$ if $f'(-1) = 0$ and $f''(-1) < 0$

   $(-1, f(-1))$ - local max

3. $(5, f(5))$ if $f'(5) = 0$ and $f''(5)$ does not exist.

   inconclusive

4. $(6, f(6))$ if $\boxed{f'(6) = 1}$ and $f''(6) > 0$.

   neither local min, nor local max

5. $(2, f(2))$ if $f'(2) = 0$ and $f''(2) = 0$

   inconclusive

Example 3. Find the absolute minimum value on $(0, \infty)$ for the function $f(x) = 4 + x + \dfrac{9}{x}$.

$$f'(x) = 1 - \frac{9}{x^2} = 0$$

$$\frac{x^2 - 9}{x^2} = 0$$

$$x^2 - 9 = 0$$

$$(x-3)(x+3) = 0$$

$$\cancel{x_1 = -3}, \quad x_2 = 3$$

not in $(0,\infty)$

$$f''(x) = (1 - 9x^{-2})'$$

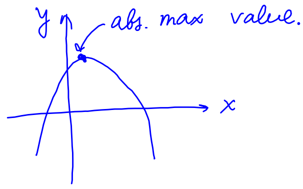
$$= -9(-2)x^{-3} = \frac{18}{x^3} \quad ; \ f''(3) = \frac{18}{27} = \frac{2}{3} > 0$$

f has the abs. min @ $x = 3$

$f(3)$ is the abs. min value

$$f(3) = 4 + 3 + \frac{9}{3} = \boxed{10}$$ abs. min value

Example 4. Find the absolute maximum and minimum, if either exists, for the following functions.



1. $f(x) = -x^2 - 6x + 9$

$f'(x) = -2x - 6 = 0$

$x = -3$

$f''(x) = -2$

$f''(-3) = -2 < 0$

f has the abs. max @ $x = -3$

$f(-3) = -(-3)^2 - 6(-3) + 9 = \boxed{18 \text{ abs. max value}}$

NO abs. min

2. $f(x) = \dfrac{-8x}{x^2+4}$.

$$f'(x) = \frac{-8(x^2+4) - (-8x)(2x)}{(x^2+4)^2} = \frac{-8x^2 - 32 + 16x^2}{(x^2+4)^2} = \frac{8x^2 - 32}{(x^2+4)^2} = 0$$

$$\frac{8x^2 - 32}{8} = \frac{0}{8}$$

$x^2 - 4 = 0$

$(x-2)(x+2) = 0$

$x_1 = 2, \quad x_2 = -2$

$$f''(x) = \frac{16x(x^2+4)^{\cancel{2}} - (8x^2-32)\,2\,\cancel{(x^2+4)}(2x)}{(x^2+4)^{\cancel{4}\,3}}$$

$$= \frac{16x(x^2+4) - (8x^2-32)(4x)}{(x^2+4)^3}$$

$$= \frac{16x^3 + 64x - 32x^3 + 128x}{(x^2+4)^3}$$

$$= \frac{-16x^3 + 192x}{(x^2+4)^3}$$

$f'(2) = 0$ and $f''(2) = \dfrac{-16(2^3) + 192(2)}{(2^2+4)^3} = .0625 > 0$

$f(2)$ is the abs. min, $f(2) = \dfrac{-8(2)}{2^2+4} = \boxed{-2 \text{ abs. min}}$

$f'(-2) = 0$ and $f''(-2) = \dfrac{-16(-2)^3 + 192(-2)}{((-2)^2+4)^3} = -.0625 < 0$

$f(-2)$ is the abs. max, $f(-2) = \dfrac{-8(-2)}{(-2)^2+4} = \boxed{2 \text{ abs. max}}$