Section 5.6 **Optimization and Modeling.**

Optimization problems are problems that involve finding the absolute maximum value or the absolute minimum value of a function.

Steps in solving optimization problems

1. Draw a picture representing the problem, if possible.
2. Select variables to represent the quantity to be maximized or minimized and all other unknowns.
3. Write an equation for the quantity to be optimized. If necessary, eliminate extra variables so that the quantity to be optimized is a function of one variable.
4. Take note of, or determine, the interval of values over which you are optimizing your function.
5. Apply the correct procedure for determining the absolute extrema on the interval you find, or are given, as learned in previous sections.
6. Answer the question posed in the problem.

Example 1. Find two numbers whose sum is 100 and whose product is a maximum

x – the 1st number

y – the 2nd number

$x+y=100$

Product $P(x)=xy$

$y=100-x$

$P(x)=x(100-x)$

$=100x-x^2$

$P'(x)=100-2x=0$

$x=50$

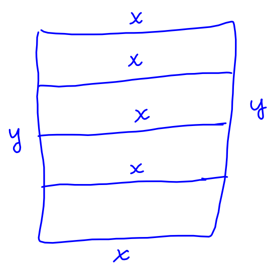
$P''(x)=-2<0$

P has the abs max @ $x=50$.

$y=100-50=50$

$\boxed{50 \text{ and } 50}$

Example 2. A farmer with 750 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of four pens?



$A = xy$ (area of the rectangle)

$5x + 2y = 750$ (total length of the fence)

$2y = 750 - 5x$

$y = \frac{750 - 5x}{2} = 375 - 2.5x$

$A = x(375 - 2.5x) = 375x - 2.5x^2$ — to be maximized

$A'(x) = 375 - 2.5(2x)$

$= 375 - 5x = 0$

$5x = 375$

$x = \frac{375}{5} = 75$

$y = 375 - 2.5x = 375 - 2.5(75) = 187.5$

$A = xy = 75(187.5) = \boxed{14062.5\,(ft^2)}$

Example 3. A company manufactures and sells x videophones per week. The weekly price-demand and cost equations are, respectively,

$$p(x) = 500 - 0.5x \quad \text{and} \quad C(x) = 20000 + 135x.$$

1. What price should the company charge for the cameras, and how many cameras should be produced to maximize the weekly revenue? What is the maximum revenue?

Revenue function $R(x) = x\,p(x)$

$= x(500 - 0.5x)$

$= 500x - 0.5x^2$ to be maximized

$R'(x) = 500 - 0.5(2x)$

$= 500 - x = 0$

$\boxed{x = 500 \text{ (cameras)}}$

$p = 500 - 0.5x = 500 - 0.5(500) = \boxed{\$250}$ to maximize the revenue

maximum revenue: $R(500) = 500(500) - 0.5(500)^2 = \boxed{\$\,125,000}$ max revenue

maximum revenue: $R(x) = x\,p(x)$

$= 500\,p(500)$

$= 500\,(250) = \boxed{\$125,000}$ max revenue

2. What is the maximum weekly profit? How much should the company charge for the cameras, and how many cameras should be produced to realize the maximum weekly profit?

$P(x) =$ Profit function $=$ Revenue $-$ cost $= R(x) - C(x)$

$= 500x - 0.5x^2 - (20000 + 135x)$

$= 500x - 0.5x^2 - 20000 - 135x$

$= -0.5x^2 + 365x - 20000$ – to be maximized

$P'(x) = -0.5(2x) + 365$

$= -x + 365 = 0$

$\boxed{x = 365 \text{ (cameras)}}$

price – $p(365) = 500 - 0.5(x)$

$= 500 - 0.5(365)$

$= \boxed{\$317.50}$ should charge for the camera to maximize the profit.

Maximum weekly profit $P(365) = -0.5(365)^2 + 365(365) - 20000 = \boxed{\$113042.5}$ max profit

Example 4. A car rental agency rents 200 cars per day at a rate of \$30 per day. For each \$1 increase in rate, 5 fewer cars are rented. At what rate should the cars be rented to produce the maximum revenue? What is the maximum revenue?

price-demand equation:

$$p - p_0 = k(x - x_0)$$

k is the slope

$$k = \frac{\Delta p}{\Delta x} = \frac{1}{-5} = -0.2$$

passes through $(\overbrace{200}^{x_0}, \overbrace{30}^{p_0})$

$$p - 30 = -0.2(x - 200)$$

$$p - 30 = -0.2x + 40$$

$$\boxed{p = -0.2x + 70}$$ price-demand equation

Revenue $R(x) = x\,p(x) = x(-0.2x + 70)$

$$= -0.2x^2 + 70x$$

$$R'(x) = -0.2(2x) + 70$$

$$= -0.4x + 70 = 0$$

$$x = \frac{70}{0.4} = 175 \text{ (cars)}$$

rate: $p(175) = -0.2x + 70$

$$= -0.2(175) + 70 = \boxed{\$35}$$

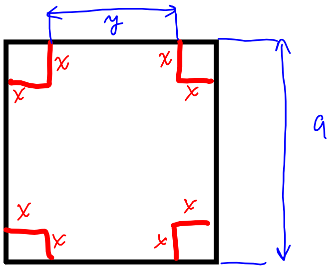
max revenue $R(175) = -0.2(175)^2 + 70(175)$

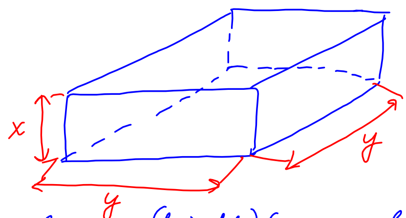
$$= x\,p(x)$$

$$= 175(35)$$

$$= \boxed{\$6125}$$ max revenue.

5. From a 9 inch x 9 inch piece of cardboard, square corners are cut out so that the sides can be folded up to form a box with no top. What should x be to maximize the volume?





Volume = (height)(area of base)

$$V = xy^2$$

$$x + y + x = 9$$
$$2x + y = 9$$
$$y = 9 - 2x$$

$V(x) = x(9-2x)^2$ to be maximized

$$V'(x) = (x)'(9-2x)^2 + x((9-2x)^2)'$$
$$= (9-2x)^2 + x\,2(9-2x)(9-2x)'$$
$$= (9-2x)^2 + 2x(9-2x)(-2)$$
$$= (9-2x)^2 - 4x(9-2x) = 0$$
$$(9-2x)(9-2x-4x) = 0$$
$$(9-2x)(9-6x) = 0$$

$9-2x=0$ or $9-6x=0$

$x = \frac{9}{2}$ or $x = \frac{9}{6} = \frac{3}{2}$

$$V''(x) = (9-2x)'(9-6x) + (9-2x)(9-6x)'$$
$$= -2(9-6x) + (-6)(9-2x)$$
$$= -18 + 12x - 54 + 12x$$
$$= 24x - 72$$

$V''\left(\frac{9}{2}\right) = 24\left(\frac{9}{2}\right) - 72 = 36 > 0$ $\cup_{9/2}$ min

$V''\left(\frac{3}{2}\right) = 24\left(\frac{3}{2}\right) - 72 = -36 < 0$ $\cap^{3/2}$ max

$$\boxed{x = \frac{3}{2}}$$