Chapter 6. **Integration**
Section 6-1 **Antiderivatives.**

Definition. Function $F(x)$ is called an antiderivative of $f(x)$ if $\boxed{F'(x) = f(x).}$
If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$\boxed{F(x) + C} \quad , \quad F'(x) = f(x).$$

where C is a constant.

Example 1. Find the general antiderivative of the function $f(x) = x^7 - 4x^2 + 2x - 6$

- $x^7 = \left(\frac{x^8}{8}\right)'$
- $-4x^2 = \left(\frac{-4x^3}{3}\right)'$
- $2x = (x^2)'$
- $-6 = (-6x)'$

$(x^8)' = 8x^7$, $\left(\frac{x^8}{8}\right)' = \frac{\cancel{8}x^7}{\cancel{8}} = x^7$

$(x^3)' = 3x^2$, $\left(\frac{-4x^3}{3}\right)' = \frac{-4(\cancel{3}x^2)}{\cancel{3}} = -4x^2$

antiderivative $F(x) = \boxed{\frac{x^8}{8} - \frac{4x^3}{3} + x^2 - 6x + C}$

The symbol

$$\boxed{\int f(x)dx}$$

$\int \_\ dx$

is called the indefinite integral and represents the family of all antiderivatives of f(x).

$$\boxed{\int f(x)dx = F(x) + C} \ \boxed{\text{if}} \ \boxed{F'(x) = f(x)}$$

The symbol $\int$ is called an **integral sign**, $f(x)$ is called the **integrand**. The symbol dx indicates that the antidifferentiation is performed with respect to the variable x. C is called the **constant of integration.**

The procedure of evaluating an integral is called integration.

Integration is the *inverse operation* of differentiation.

Table of indefinite integrals.

1. $\int a\ dx = ax + C$, a is a constant
2. $\displaystyle\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$
3. $\displaystyle\int \frac{dx}{x} = \ln|x| + C$
4. $\int e^x dx = e^x + C$

$\int x\, dx = \frac{x^2}{2} + C$

$\int x^2 dx = \frac{x^3}{3} + C$

$\int x^3 = \frac{x^4}{4} + C$

Properties of indefinite integrals.
For k a constant,

1. $\int kf(x)dx = k\int f(x)dx$
2. $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$

Example 2. Find the indefinite integral.

1. $\int 5dx = \boxed{5x + C}$

2. $\int \frac{2}{5}t^2 dt = \frac{2}{5}\int t^2 dt = \frac{2}{5}\,\frac{t^3}{3} + C = \boxed{\frac{2}{15}t^3 + C}$

3. $\int \frac{5}{t^3}dt = \int 5t^{-3}dt = 5\int t^{-3}dt = \frac{5t^{-3+1}}{-3+1} + C$
$$= \boxed{\frac{5t^{-2}}{-2} + C}$$

4. $\int (x^2 - x + 4)dx = \int x^2 dx - \int x\,dx + \int 4\,dx$
$$= \boxed{\frac{x^3}{3} - \frac{x^2}{2} + 4x + C}$$

5. $\int x^2(x^3 - 2x + 1)dx = \int (x^5 - 2x^3 + x^2)dx = \int x^5 dx - 2\int x^3 dx + \int x^2 dx$
$$= \frac{x^{5+1}}{5+1} - 2\,\frac{x^{3+1}}{3+1} + \frac{x^3}{3} + C$$
$$= \frac{x^6}{6} - \frac{\cancel{2}x^4}{\cancel{4}\,2} + \frac{x^3}{3} + C = \boxed{\frac{x^6}{6} - \frac{x^4}{2} + \frac{x^3}{3} + C}$$

6. $\int (x^2 - 3)(1 + x^3)dx = \int (x^2 + x^5 - 3 - 3x^3)dx = \int x^2 dx + \int x^5 dx - \int 3dx - 3\int x^3 dx$
$$= \frac{x^3}{3} + \frac{x^{5+1}}{5+1} - 3x - 3\,\frac{x^{3+1}}{3+1} + C$$
$$= \boxed{\frac{x^3}{3} + \frac{x^6}{6} - 3x - 3\frac{x^4}{4} + C}$$

7. $\int \frac{5}{\sqrt[3]{t}}dt = \int 5t^{-1/3}dt = 5\int t^{-1/3}dt = 5\,\frac{t^{-1/3+1}}{-1/3+1} + C$
$$= \frac{5t^{2/3}}{2/3} + C = 5\cdot\frac{3}{2}\,t^{2/3} + C = \boxed{\frac{15}{2}t^{2/3} + C}$$

8. $\int \frac{1 - y^2}{y^3}dy = \int \left(\frac{1}{y^3} - \frac{\cancel{y^2}}{y^{\cancel{3}}}\right)dy = \int \left(y^{-3} - \frac{1}{y}\right)dy$
$$= \int y^{-3}dy - \int \frac{1}{y}dy = \frac{y^{-3+1}}{-3+1} - \ln|y| + C$$
$$= \boxed{\frac{y^{-2}}{-2} - \ln|y| + C}$$

9. $\int \left(\frac{2}{\sqrt[4]{x}} - \sqrt[5]{x^3} \right) dx = \int (2x^{-1/4} - x^{3/5})\,dx = 2\int x^{-1/4}\,dx - \int x^{3/5}\,dx$

$= 2\,\frac{x^{-1/4+1}}{-1/4+1} - \frac{x^{3/5+1}}{3/5+1} + C$

$= 2\,\frac{x^{3/4}}{3/4} - \frac{x^{8/5}}{8/5} + C$

$= 2\,\frac{4}{3}\,x^{3/4} - \frac{5}{8}\,x^{8/5} + C$

$= \boxed{\frac{8}{3}x^{3/4} - \frac{5}{8}x^{8/5} + C}$

10. $\int \frac{4e^x - 3x^2}{2}\,dx$

$= \frac{1}{2}\int (4e^x - 3x^2)\,dx = \frac{\cancel{4}\,2}{\cancel{2}}\int e^x\,dx - \frac{3}{2}\int x^2\,dx$

$= 2e^x - \frac{\cancel{3}}{2}\,\frac{x^3}{\cancel{3}} + C = \boxed{2e^x - \frac{1}{2}x^3 + C}$

Example 3. The marginal average cost of producing x sports watches is given by

$$\bar{C}'(x) = -\frac{1000}{x^2}, \quad \bar{C}(100) = 25 \text{ – condition}$$

where $\bar{C}(x)$ is the average cost in dollars. Find the average cost function and the cost function.

the average cost: $\bar{C}(x) = \int \bar{C}'(x)\,dx$

$= \int \left(-\frac{1000}{x^2}\right) dx = -1000\int x^{-2}\,dx$

$= -1000\,\frac{x^{-2+1}}{-2+1} + C = -1000\,\frac{x^{-1}}{-1} + C$

$= 1000x^{-1} + C$

condition $\bar{C}(100) = 25$

$\bar{C}(100) = 1000(100)^{-1} + C$

$= 10 + C = 25$

$\boxed{C = 15}$

average cost $\boxed{\bar{C}(x) = 1000x^{-1} + 15}$

cost function $C(x) = x\,\bar{C}(x)$

$= x(1000x^{-1} + 15)$

$= \boxed{1000 + 15x}$