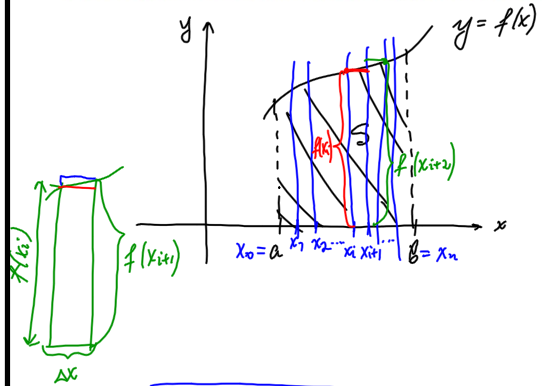
Problem. Find the area of the region S that lies under the curve $y = f(x)$ from a to b.



1. We partition the interval $[a,b]$ into n subintervals of equal length $\Delta x = \frac{b-a}{n}$

2) Partition points:

$x_0 = a$

$x_1 = x_0 + \Delta x = a + \frac{b-a}{n}$

$x_2 = x_0 + 2\Delta x = a + 2\frac{b-a}{n}$

$x_3 = x_0 + 3\Delta x = a + 3\frac{b-a}{n}$

$\cdots$

$x_n = b$

3) draw the lines $x = x_0$, $x = x_1$, $x = x_2, \ldots$, $x = x_n$

the lines brake the region S into strips.

4) approximate each strip by a rectangle

the area of a rectangle $=$ (width)$\times$(height)

$A = f(x_i)\Delta x$ or $A = f(x_{i+1})\Delta x$, $i = 0, 1, 2, \ldots$

5) add the areas of rectangles

Then, we have

$\boxed{\textbf{Left Hand Sum (LHS)}}$ $L_n = \overbrace{f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x}^{n \text{ terms}} = \sum_{k=1}^{n} f(x_{k-1})\Delta x$

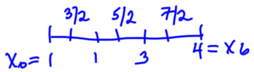
$\boxed{\textbf{Right Hand Sum (RHS)}}$ $R_n = \underbrace{f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x}_{n \text{ terms}} = \sum_{k=1}^{n} f(x_k)\Delta x$

Then we can approximate the area of the region S using one of these sums.

Theorem. If $f(x) > 0$ and either increasing on $[a, b]$ or decreasing on $[a, b]$, then its LHS and RHS approach the same real number as $n \to \infty$.

Example 1. Approximate the area under the curve $y = x^2 + 3x - 2$ from 1 to 4 using

1. RHS R_6 $\quad n = 6$



1) Divide $[1,4]$ into 6 subintervals of equal length $\Delta x = \frac{4-1}{6} = \frac{1}{2}$

2) Partition points:

$x_0 = 1$

$x_1 = x_0 + \Delta x = 1 + \frac{1}{2} = \frac{3}{2} = 1.5$

$x_2 = x_0 + 2\Delta x = 1 + 2\frac{1}{2} = 2$

$x_3 = x_0 + 3\Delta x = 1 + 3\frac{1}{2} = \frac{5}{2} = 2.5$

$x_4 = x_0 + 4\Delta x = 1 + 4\frac{1}{2} = 3$

$x_5 = x_0 + 5\Delta x = 1 + 5\frac{1}{2} = \frac{7}{2} = 3.5$

$x_6 = 4$

3) $f(x_0) = f(1) = 2$

$f(x_1) = f(1.5) = 4.75$

$f(x_2) = f(2) = 8$

$f(x_3) = f(2.5) = 11.75$

$f(x_4) = f(3) = 16$

$f(x_5) = f(3.5) = 20.75$

$f(x_6) = f(4) = 26$

2. LHS L_6

4) $R_6 = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x$

$= 4.75 \cdot \frac{1}{2} + 8 \cdot \frac{1}{2} + 11.75 \cdot \frac{1}{2} + 16 \cdot \frac{1}{2} + 20.75 \frac{1}{2} + 26 \frac{1}{2}$

$= \boxed{43.625}$

$L_6 = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x$

$= 2 \cdot \frac{1}{2} + 4.75 \cdot \frac{1}{2} + 8 \cdot \frac{1}{2} + 11.75 \cdot \frac{1}{2} + 16 \cdot \frac{1}{2} + 20.75 \cdot \frac{1}{2}$

$= \boxed{31.625}$

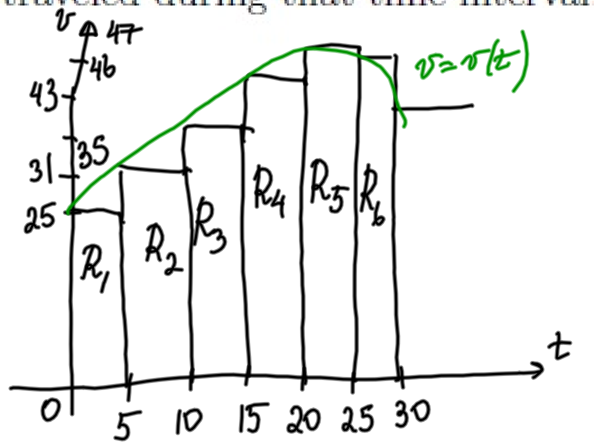
The distance problem.

Finding the distance traveled by an object over a certain period of time is simple if the velocity remains constant (distance = velocity$\times$ time). But, if the velocity varies over time, it is not as easy.

Example 2. The table below shows the velocity (in ft/s) of an object every five seconds over a 30 second time interval. Estimate the total distance the object travels over the 30 second time interval by using the velocities at the beginning of the time intervals.

Time (s)	0	5	10	15	20	25	30
Velocity (ft/s)	25	31	35	43	47	46	41

If we sketch a graph of the velocity function and draw rectangles whose heights are the initial velocities for each time interval, then the area of each rectangle can be interpreted as the distance traveled during that time interval.



distance traveled $\approx$

$$A(R_1)+A(R_2)+A(R_3)+A(R_4)+A(R_5)+A(R_6)$$

$$= 5(25)+5(31)+5(35)+5(43)+5(47)+5(46)$$

$$= \boxed{1{,}135\ (ft)}$$

Example 3. A model rocket has upward velocity $v(t) = 20t^2$ ft/s, t seconds after launch. Use the interval $[0, 5]$ with $n = 5$ and equal subintervals to compute the following approximations of the distance the rocket traveled. (Round your answers to two decimal places.)

1. RHS R_5

Divide $[0,5]$ into 5 intervals of equal lengths

$$\Delta x = \frac{5-0}{5} = 1$$

Partition points:

$x_0 = 0$	$f(0) = 0$
$x_1 = 1$	$f(1) = 20$
$x_2 = 2$	$f(2) = 80$
$x_3 = 3$	$f(3) = 180$
$x_4 = 4$	$f(4) = 320$
$x_5 = 5$	$f(5) = 500$

$$R_5 = f(1)1 + f(2)1 + f(3)1 + f(4)1 + f(5)1 = 20 + 80 + 180 + 320 + 500 = \boxed{1{,}100}$$

2. LHS L_5

$$L_5 = f(0)1 + f(1)1 + f(2)1 + f(3)1 + f(4)1 = 0 + 20 + 80 + 180 + 320 = \boxed{600}$$