$\int_a^b f(x)\,dx$ = a number

Section 6.4 The Definite Integral.

We partition $[a, b]$ into n subintervals of equal length $\Delta x = (b - a)/n$ with endpoints

$$a = x_0 < x_1 < x_2 < \dots < x_n = b.$$

Then, we have

$$\textbf{LHS} \quad L_n = f(x_0)\Delta x + f(x_1)\Delta x + \dots + f(x_{n-1})\Delta x = \sum_{k=1}^{n} f(x_{k-1})\Delta x$$

$$\textbf{RHS} \quad R_n = f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x = \sum_{k=1}^{n} f(x_k)\Delta x$$

$$\textbf{Riemann sum} \quad S_n = f(c_1)\Delta x + f(c_2)\Delta x + \dots + f(c_n)\Delta x = \sum_{k=1}^{n} f(c_k)\Delta x$$

here c_k belongs to the subinterval $[x_{k-1}, x_k]$.
Theorem. If f is continuous function on $[a, b]$, then the Riemann sums approach to a real number limit I as $n \to \infty$.

Definition of a definite integral. Let f be continuous function on $[a, b]$. Then the limit I of Riemann sums for f on $[a, b]$ is called the **definite integral** of f from a to b and is denoted by

$$\int_a^b f(x)dx$$

The **integrand** is $f(x)$, the **lower limit of integration** is a, and the **upper limit of integration** is b.

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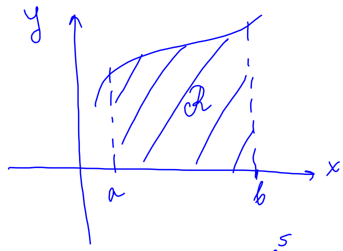
if $f(x) \geq 0$ on $[a, b]$, then

$\int_a^b f(x)dx$ = area below the graph of $f(x)$ between $x=a$, $x=b$, above the x-axis.



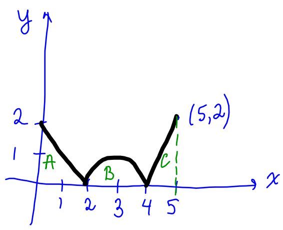
$\boxed{\int_a^b f(x)dx = \text{area } (R)}$

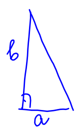
Example 0. Find $\int_0^5 f(x)\,dx$ = area (A) + area (B) + area (C)

$= \frac{1}{2}(2)(2) + \frac{1}{2}\pi(1)^2 + \frac{1}{2}(1)(2)$

$= 2 + 1 + \frac{\pi}{2}$

$= \boxed{3 + \frac{\pi}{2}}$





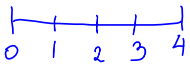
$A = \frac{1}{2}ab$

$A = \pi r^2$

Example 1. Partition interval [0,4] into four subintervals of equal length. For the function $f(x) = x^2 - 4x + 6$ calculate: | Approximate $\int_0^4 (x^2 - 4x + 6)\,dx$ using:

1. the left-hand sum L_4

$\Delta x = \frac{4-0}{4} = 1$



Partition points

$x_0 = 0,\ x_1 = 0 + \Delta x = 1,\ x_2 = 0 + 2\Delta x = 2,\ x_3 = 0 + 3\Delta x = 3,\ x_4 = 4$

$f(0) = 6$

$f(1) = 3$

$f(2) = 2$

$f(3) = 3$

$f(4) = 6$

$L_4 = f(0)\Delta x + f(1)\Delta x + f(2)\Delta x + f(3)\Delta x \overset{\Delta x = 1}{=} 6 + 3 + 2 + 3 = \boxed{14}$

2. the right-hand sum $R_4 = f(1)\Delta x + f(2)\Delta x + f(3)\Delta x + f(4)\Delta x \overset{\Delta x = 1}{=} 3 + 2 + 3 + 6 = \boxed{14}$

3. the Riemann sum S_4 if $c_1 = .4$, $c_2 = 1.2$, $c_3 = 2.3$, and $c_4 = 3.6$.

$S_4 = f(c_1)\Delta x + f(c_2)\Delta x + f(c_3)\Delta x + f(c_4)\Delta x$

$\overset{\Delta x = 1}{=} f(0.4) + f(1.2) + f(2.3) + f(3.6)$

$f(0.4) = 4.56$

$f(1.2) = 2.64$

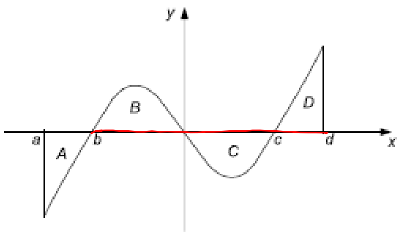
$f(2.3) = 2.09$

$f(3.6) = 4.56$

$S_4 = 4.56 + 2.64 + 2.09 + 4.56 = \boxed{13.85}$

Because area is a positive quantity, the definite integral represents the cumulative sum of the signed areas between the graph of f and the x-axis from $x = a$ to $x = b$, where the areas above the x-axis are counted positively and the areas below the x-axis are counted negatively.

Example 2. Calculate definite integrals by reffering to the figure



if area $A = 1.22$, area $B = 2.3$, area $C = 2.5$, and area $D = 1.6$.

2

1. $\int_a^b f(x)dx = -\text{area}(A) = \boxed{-1.22}$

2. $\int_a^c f(x)dx = -\text{area}(A) + \text{area}(B) - \text{area}(C)$
$= -1.22 + 2.3 - 2.5 = \boxed{-1.42}$

3. $\int_b^d f(x)dx = \text{area}(B) - \text{area}(C) + \text{area}(D)$
$= 2.3 - 2.5 + 1.6 = \boxed{1.4}$

Properties of the definite integral

1. $\int_a^a f(x)dx = 0$
2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
3. $\int_a^b c\,dx = c(b-a)$, where c is a constant
4. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, where c is a constant
5. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
6. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $a < c < b$

Example 3. Given that $\int_1^4 x\,dx = 4$, $\int_1^4 x^2dx = -2$, and $\int_4^6 x^2dx = 2$.

Find

1. $\int_1^1 (2x - 3x^2)dx = 0$

2. $\int_1^6 x^2dx \overset{\text{Property 6}}{=} \int_1^4 x^2dx + \int_4^6 x^2dx$

   $= -2 + 2 = \boxed{0}$

3. $\int_1^4 x(2x - 3)dx = \int_1^4 x(2x-3)\,dx = \int_1^4 (2x^2 - 3x)\,dx$

   $= \int_1^4 2x^2\,dx - \int_1^4 3x\,dx$

   $= 2\underbrace{\int_1^4 x^2dx}_{-2} - 3\underbrace{\int_1^4 x\,dx}_{4}$

   $= 2(-2) - 3(4) = -4 - 12 = \boxed{-16}$