Section 6.5. **The Fundamental Theorem of Calculus.**

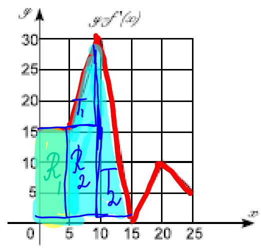
Theorem. Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.

2. $\boxed{\int_a^b f(x)dx = F(x)\Big]_a^b = F(b) - F(a)}$, where F is an antiderivative of f.

$\boxed{\int_a^b f'(x)dx = f(b) - f(a)}$

Example 1. Use the graph of $f'(x)$, given below, and the fact that $\boxed{f(0) = 10}$ to compute the following values.



1. $f(5)$

$\int_0^5 f'(x)dx = f(5) - \underbrace{f(0)}_{10}$, where $\int_0^5 f'(x)dx = 75$

$\int_0^5 f'(x)dx = \text{area }(R_1) = 5(15) = 75$

$75 = f(5) - 10$

$f(5) = 75 + 10$

$= \boxed{85}$

2. $f(15)$

$\int_0^{15} f'(x)dx = f(15) - f(0)$

$262.5 = f(15) - 10$

$\int_0^{15} f'(x)dx = \text{area }(R_1) + \text{area }(R_2) + \text{area }(T_1) + \text{area }(T_2)$

$= 75 + 75 + \frac{1}{2}5(30-15) + \frac{1}{2}(5)(30) = 262.5$

$f(15) = 262.5 + f(0) = 262.5 + 10 = \boxed{272.5}$

Example 2. If $f(1) = 14$, f' is continuous, and $\int_1^6 f'(x)dx = 25$, what is the value of $f(6)$?

$\int_1^6 f'(x)dx = 25$

$\underbrace{\int_1^6 f'(x)dx}_{25} = f(6) - \underbrace{f(1)}_{14}$

$f(6) = 25 + 14 = \boxed{39}$

Example 3. Evaluate the following definite integrals:

1. $$\int_1^2 \left(\sqrt[4]{x^3} - \frac{1}{6x}\right) dx = \int_1^2 \left(x^{3/4} - \frac{1}{6x}\right) dx = \int_1^2 x^{3/4}\,dx - \frac{1}{6}\int_1^2 \frac{1}{x}\,dx$$
$$= \frac{x^{3/4+1}}{\frac{3}{4}+1}\Big]_1^2 - \frac{1}{6}\ln|x|\Big]_1^2 = \frac{4}{7}x^{7/4}\Big]_1^2 - \frac{1}{6}\left(\ln|2| - \cancel{\ln|1|}^{\,0}\right)$$
$$= \frac{4}{7}\left(2^{7/4} - \cancel{1^{7/4}}^{\,1}\right) - \frac{1}{6}\ln 2$$
$$= \boxed{\frac{4}{7}\left(2^{7/4}-1\right) - \frac{1}{6}\ln 2}$$

2. $$\frac{1}{3}\int_0^1 3x^2 e^{x^3}\,dx \quad \left|\begin{array}{l} u = x^3 \\ du = 3x^2\,dx \end{array}\right| = \frac{1}{3}\int_{x=0}^{x=1} e^u\,du = \frac{1}{3}e^u\Big]_{x=0}^{x=1}$$
$$= \frac{1}{3}e^{x^3}\Big]_{x=0}^{x=1} = \frac{1}{3}\left(e^{1^3} - e^{0^3}\right)$$
$$= \frac{1}{3}\left(e^1 - e^0\right)$$
$$= \boxed{\frac{1}{3}(e-1)}$$

3. $$\int_0^A \frac{x}{\sqrt{4-x}}\,dx, \text{ where } A > 0.$$ A is a constant $$\left|\begin{array}{l} u = 4-x \\ du = -dx \\ x = 4-u \end{array}\right|$$
$$= -\int_{x=0}^{x=A} \frac{4-u}{\sqrt{u}}\,du = -\int_{x=0}^{x=A}\left(\frac{4}{\sqrt{u}} - \frac{u}{\sqrt{u}}\right)du = -\int_{x=0}^{x=A}\left(4u^{-1/2} - u^{1/2}\right)du$$
$$= -\left(\frac{4u^{-1/2+1}}{-\frac{1}{2}+1} - \frac{u^{1/2+1}}{1/2+1}\right)\Big]_{x=0}^{x=A} = -\left(4(2)u^{1/2} - \frac{2}{3}u^{3/2}\right)\Big]_{x=0}^{x=A}$$
$$= -\left(8(4-x)^{1/2} - \frac{2}{3}(4-x)^{3/2}\right)\Big]_{x=0}^{x=A} = -\left(8(4-A)^{1/2} - \frac{2}{3}(4-A)^{3/2}\right) + \left(8(4)^{1/2} - \frac{2}{3}(4)^{3/2}\right)$$
$$= \boxed{-\left(8(4-A)^{1/2} - \frac{2}{3}(4-A)^{3/2}\right) + \underbrace{\left(8(2) - \frac{2}{3}(8)\right)}_{32/3}}$$

4. $$\frac{1}{2}\int_B^0 \frac{2(x-1)}{x^2-2x+3}\,dx = \left|\begin{array}{l} u = x^2 - 2x + 3 \\ du = (2x-2)\,dx \\ \quad = 2(x-1)\,dx \end{array}\right|$$
$$= \frac{1}{2}\int_{x=B}^{x=0} \frac{1}{u}\,du = \frac{1}{2}\ln|u|\Big]_{x=B}^{x=0} = \frac{1}{2}\ln|x^2 - 2x + 3|\Big]_{x=B}^{x=0}$$
$$= \boxed{\frac{1}{2}\ln|3| - \frac{1}{2}\ln|B^2 - 2B + 3|}$$

Finding Aproximations of Definite Integrals on the Calculator. If f is a continuous func- b tion on $[a, b]$, you can estimate the value of the definite integral $\int_a^b f(x)dx$ by the following command from your home screen

$$\boxed{\text{MATH}} \downarrow 9 : \text{fnInt} \implies \text{fnInt}(f(x), x, a, b)$$

Example 3. Evaluate the definite integral

$$\int_{1.7}^{3.5} x \ln x \, dx \approx \boxed{4.566}$$

to three decimal places.

Example 4. A managerial service determines that the rate of increase in maintenance costs (in dollars per year) for a particular apartment complex is given approximately by

$$M'(x) = f(x) = 90x^2 + 5000$$

where x is the age of the apartment complex in years and $M(x)$ is the total (accumulated) cost of maintenance for x years. Write a definite integral that will give the total maintenance costs from the end of the second year to the end of the seventh year after the apartment complex was built, and evaluate the integral.

total maintenance costs at the end of the Ath year is

$$\int_0^A M'(x)\,dx$$

$$\text{costs} = \int_2^7 M'(x)dx = \int_2^7 (90x^2 + 5000)\,dx = \boxed{35,050}$$

Average value of a continuous function f over $[a, b]$ is

$$\boxed{f\text{ave} = \frac{1}{b-a}\int_a^b f(x)dx}$$

Example 5. Find the average value of the function $f(x) = \sqrt[3]{x}$ over the interval $[1, 8]$.

$$f_{ave} = \frac{1}{8-1}\int_1^8 \sqrt[3]{x}\,dx = \frac{1}{7}\int_1^8 x^{1/3}\,dx = \frac{11.25}{7} = 1.61 = \boxed{\frac{45}{28}}$$