Section 8.1. **Functions of Several Variables.**

An equation of the form $\boxed{z = f(x, y)}$ describes a function of two variables if, for each permissible ordered pair (x, y), there is one and only one value of z determined by $f(x, y)$.

- The variables x and y are **independent variables**, and the variable z is a **dependent variable**.
- The set of all ordered pairs of permissible values of x and y is the **domain** of the function, and the set of all corresponding values $f(x, y)$ is the **range** of the function.

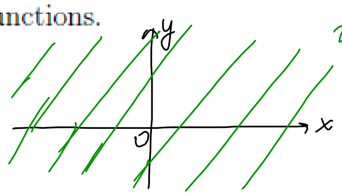
We will assume that the domain of a function $z = f(x, y)$ is the set of all ordered pairs of real numbers (x, y) such that $f(x, y)$ is also a real number.

Example 1. Let $f(x, y) = 2x^2 - 3xy + 4\sqrt{y}$. Find $\cancel{f(4,-2)}$.

1. $f(0, 4) = 2(0^2) - 3(0)(4) + 4\sqrt{4} = \boxed{8}$
   $x=0$
   $y=4$
2. $f(2, 0) = 2(2^2) - 3(2)(0) + 4\sqrt{0} = \boxed{8}$
   $x=2$
   $y=0$
3. $f(-2, 9) = 2(-2)^2 - 3(-2)(9) + 4\sqrt{9} = \boxed{74}$
   $x=-2$
   $y=9$

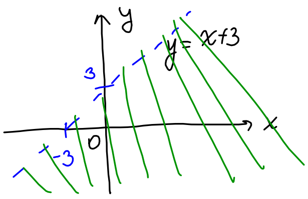
Example 2. Find the domain of each of the following functions.

1. $f(x, y) = 2x - 3y$
   Domain: $\boxed{\begin{array}{l} -\infty < x < \infty \\ -\infty < y < \infty \end{array}}$



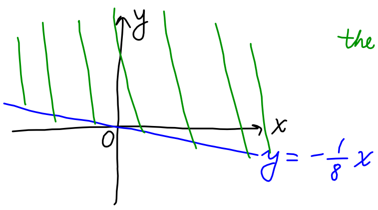
whole plane

2. $f(x, y) = \ln(x - y + 3)$
   Domain: $x - y + 3 > 0$
   solve for y:
   Domain: $\boxed{y < x+3}$



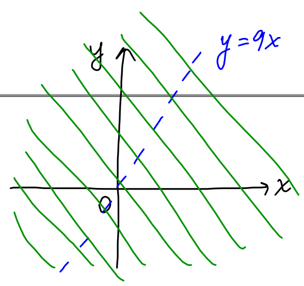
the line $y = x+3$ is NOT included in the domain

3. $f(x, y) = \sqrt[4]{x + 8y}$
   Domain $x + 8y \geq 0$
   solve for y: $8y \geq -x$
   Domain $\boxed{y \geq -\frac{1}{8}x}$



the line $y = -\frac{1}{8}x$ is included in the domain

4. $f(x, y) = \dfrac{x^2 - 4}{9x - y}$
   Domain: $9x - y \neq 0$
   $\boxed{y \neq 9x}$



the line $y = 9x$ is NOT included in the domain

Example 3. A company manufactures 10- and 13- speed bicycles. The weekly demand and cost equations are

$$p = 230 - 9x + y$$
$$q = 130 + x - 4y$$
$$C(x, y) = 200 + 80x + 30y \text{ — cost equation}$$

where \$$p$ is the price of a 10-speed bicycle, \$$q$ is the price of a 13-speed bicycle, x is the weekly demand for 10-speed bicycle, y is the weekly demand for 13-speed bicycle, $C(x, y)$ is the cost function. Find the weekly revenue function $R(x, y)$ and the weekly profit function $P(x, y)$. Evaluate $R(10, 15)$ and $P(10, 15)$.

Revenue: $R(x,y) = xp + yq = x(230 - 9x + y) + y(130 + x - 4y)$

$= 230x - 9x^2 + xy + 130y + xy - 4y^2$

$\boxed{R(x,y) = 230x - 9x^2 + 2xy + 130y - 4y^2}$

Profit: $\boxed{P(x,y) = R(x,y) - C(x,y) = 230x - 9x^2 + 2xy + 130y - 4y^2 - 200 - 80x - 30y}$

$R(10, 15) = 230(10) - 9(10^2) + 2(10)(15) + 130(15) - 4(15^2)$

Cobb-Douglas production function

$$\boxed{f(x, y) = kx^m y^n}$$

where k, m, and n are positive constants with $m + n = 1$. Economists use this function to describe the number of units $f(x, y)$ produced from the utilization of x units of labor and y units of capital (for equipment such as tools, machinery, buildings and so on).

Example 5. The Cobb-Douglas production function for a petroleum company is given by

$$f(x, y) = 10x^{0.4}y^{0.6}$$

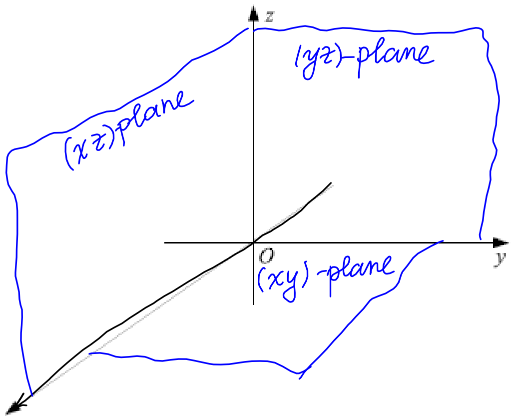
where x is the utilization of labor and y is the utilization of capital. If the company uses 1250 units of labor and 1700 units of capital, how many units of petroleum will be produced?

$x = 1250, \ y = 1700$

$f(1250, 1700) = 10(1250^{0.4})(1700^{0.6}) = \boxed{15032.57}$

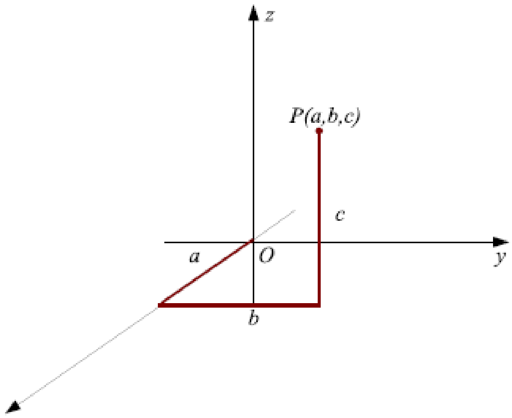
Three-dimensional coordinate system.

We first choose a fixed point O (the origin) and tree directed lines through O that are perpendicular to each other, called the **coordinate axes** and labeled the x-axis, y-axis, and z-axis. Usually we think of the x and y-axes as being horizontal and z-axis as being vertical.



The three coordinate axes determine the three **coordinate planes**. The xy-plane contains the x- and y-axes and its equation is $z = 0$, the xz-plane contains the x- and z-axes and its equation is $y = 0$. The yz-plane contains the y- and z-axes and its equation is $x = 0$. These three coordinate planes divide space into eight parts called **octants**. The **first octant** is determined by positive axes.

Take a point P in space, let a be directed distance from yz-plane to P, b be directed distance from xz-plane to P, and c be directed distance from xy-plane to P.



We represent the point P by the ordered triple (a, b, c) of real numbers, and we call a, b, and c the **coordinates** of P.