Section 8.2. **Partial Derivatives.**

First-Order Partial Derivatives.

Given $z = f(x, y)$, we define the first-order partial derivatives of f with respect to x AND y.

$\boxed{\dfrac{\partial f}{\partial x} = f_x(x, y) = f_x}$ = the first-order partial derivative of f with respect to x (**y is constant**)

$\boxed{\dfrac{\partial f}{\partial y} = f_y(x, y) = f_y}$ = the first-order partial derivative of f with respect to y (**x is constant**)

Example 1. Find the first-order partial derivatives of the following functions:

1. $f(x, y) = x^3 - y^2$

$f_x = 3x^2 - 0 = 3x^2$

$f_y = 0 - 2y = -2y$

2. $f(x, y) = 3x^2 + xy^3 + 4y$

$f_x = 6x + y^3 + 0 = 6x + y^3$

$f_y = x(3y^2) + 4 = 3xy^2 + 4$

3. $f(x, y) = \dfrac{x^4 + 7y}{5x} = \frac{1}{5}(x^4 + 7y)x^{-1} = \frac{1}{5}(x^4 \cdot x^{-1} + 7y\,x^{-1})$

$= \frac{1}{5}(x^3 + 7y\,x^{-1})$

$f_x = \frac{1}{5}(3x^2 + 7y(-1)x^{-2}) = \frac{1}{5}(3x^2 - 7y\,x^{-2})$

$f_y = \frac{1}{5}(0 + 7x^{-1}) = \frac{7}{5}x^{-1}$

4. $f(x, y) = e^{2x - y^2} = e^{2x} \cdot e^{-y^2}$

$\boxed{(e^{f(x)})' = e^{f(x)} f'(x)}$

$f_x = (e^{2x})'_x (e^{-y^2}) = 2e^{2x}e^{-y^2}$

$f_y = e^{2x}(e^{-y^2})'_y = e^{2x}e^{-y^2}(-y^2)'_y = e^{2x}e^{-y^2}(-2y)$

5. $f(x, y) = \ln(2x^2 + xy - y^5)$

$$\boxed{[\ln(f(x))]' = \frac{f'(x)}{f(x)}}$$

$$f_x = \frac{(2x^2+xy-y^5)'_x}{2x^2+xy-y^5} = \frac{4x+y}{2x^2+xy-y^5}$$

$$f_y = \frac{(2x^2+xy-y^5)'_y}{2x^2+xy-y^5} = \frac{x-5y^4}{2x^2+xy-y^5}$$

6. $f(x, y) = (x^3 + xy^2 + 3y)^6$

$$\boxed{[(u(x))^n]' = n(u(x))^{n-1}\,u'(x)}$$

$$f_x = 6(x^3+xy^2+3y)^5\,(x^3+xy^2+3y)'_x = 6(x^3+xy^2+3y)^5(3x^2+y^2)$$

$$f_y = 6(x^3+xy^2+3y)^5\,(x^3+xy^2+3y)'_y = 6(x^3+xy^2+3y)^5(2xy+3)$$

Example 2. The productivity of a certain third-world country is given approximately by the function $\boxed{f(L, K) = 10L^{0.75}K^{0.25}}$ with the utilization of L units of labor and K units of capital.

1. Find $f_L(L, K)$ and $f_K(L, K)$.

$$f_L = 10(0.75)L^{0.75-1}K^{0.25} = 7.5\,L^{-0.25}K^{0.25}$$ – marginal productivity of labor

$$f_K = 10\,L^{0.75}(0.25)K^{0.25-1} = 2.5\,L^{0.75}K^{-0.75}$$ – marginal productivity of capital

2. If the country is now using 600 units of labor and 100 units of capital, find the **marginal productivity of labor** and the **marginal productivity of capital**.

$L = 600,\ K = 100$

$$f_L(600, 100) = 7.5(600^{-0.25})(100^{0.25}) \approx 4.79$$

$$f_K(600, 100) = 2.5(600^{0.75})(100^{-0.75}) \approx 9.58$$

Second-Order Partial Derivatives

Given $z = f(x, y)$, there are four second-order partial derivatives.

$$\boxed{\frac{\partial^2 f}{\partial x^2} = f_{xx}(x, y) = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)}$$

$$\boxed{\frac{\partial^2 f}{\partial y^2} = f_{yy}(x, y) = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)}$$

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = f_{xy}(x, y) = f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)}$$

$f_{xy} = f_{yx}$

$$\boxed{\frac{\partial^2 f}{\partial y \partial x} = f_{yx}(x, y) = f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)}$$

Note: In subscript notation, differentiate from left to right.

For almost all functions, $\boxed{\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}}$

Example 3. Find all second-order partial derivatives of the following functions.

1. $f(x, y) = \dfrac{x^2}{y^3} = x^2 y^{-3}$

$f_x = 2x y^{-3}$

$f_y = x^2 (-3) y^{-3-1} = -3x^2 y^{-4}$

$f_{xx} = (2x y^{-3})'_x = 2y^{-3}$

$f_{yy} = (-3x^2 y^{-4})'_y = -3x^2(-4) y^{-4-1} = 12x^2 y^{-5}$

$f_{xy} = (2x y^{-3})'_y$ | $= (-3x^2 y^{-4})'_x$

$= 2x(-3) y^{-3-1}$ | $= -3(2x) y^{-4}$

$= -6x y^{-4}$ | $= -6x y^{-4}$

$$\boxed{\left(e^{u(x)}\right)' = e^{u(x)}\, u'(x)}$$

2. $f(x, y) = xye^{xy}$

$$f_x \overset{\text{Product Rule}}{=} (xy)'_x e^{xy} + xy\left(e^{xy}\right)'_x$$

$$= ye^{xy} + xye^{xy}\underbrace{(xy)'_x}_{y} = \boxed{ye^{xy} + xy^2e^{xy} = f_x}$$

$$f_y \overset{\text{Product Rule}}{=} (xy)'_y\, e^{xy} + xy\left(e^{xy}\right)'_y$$

$$= xe^{xy} + xy\, e^{xy}\underbrace{(xy)'_y}_{x} = \boxed{xe^{xy} + x^2ye^{xy} = f_y}$$

$$\boxed{\left(e^{xy}\right)'_x = ye^{xy}} \quad \boxed{\left(e^{xy}\right)'_y = xe^{xy}}$$

$$f_{xx} = (f_x)'_x = \left(ye^{xy} + xy^2e^{xy}\right)'_x$$

$$= y\underbrace{\left(e^{xy}\right)'_x}_{ye^{xy}} + \underbrace{(xy^2)'_x}_{y^2} e^{xy} + xy^2\underbrace{\left(e^{xy}\right)'_x}_{ye^{xy}}$$

$$= y^2e^{xy} + y^2e^{xy} + xy^3e^{xy} = \boxed{2y^2e^{xy} + xy^3e^{xy} = f_{xx}}$$

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$$f_{xy} = (f_x)'_y = (f_y)'_x = \left(ye^{xy} + xy^2e^{xy}\right)'_y$$

$$= \underbrace{(y)'_y}_{1} e^{xy} + y\underbrace{\left(e^{xy}\right)'_y}_{xe^{xy}} + \underbrace{(xy^2)'_y}_{2xy} e^{xy} + xy^2\underbrace{\left(e^{xy}\right)'_y}_{xe^{xy}}$$

$$= e^{xy} + xye^{xy} + 2xye^{xy} + x^2y^2e^{xy}$$

$$= \boxed{e^{xy} + 3xye^{xy} + x^2y^2e^{xy} = f_{xy}}$$

$$f_{yy} = (f_y)'_y = \left(xe^{xy} + x^2ye^{xy}\right)'_y$$

$$= x\underbrace{\left(e^{xy}\right)'_y}_{xe^{xy}} + \underbrace{(x^2y)'_y}_{x^2} e^{xy} + x^2y\underbrace{\left(e^{xy}\right)'_y}_{xe^{xy}}$$

$$= x^2e^{xy} + x^2e^{xy} + x^3ye^{xy}$$

$$= \boxed{2x^2e^{xy} + x^3ye^{xy} = f_{yy}}$$

2. $\cancel{f(x, y) = xye^{-xy}}$

Functions can have more than just one or two independent variables. For instance, a function of three independent variables could be $w = f(x, y, z)$. Here w would have three partial derivatives, one for each independent variable, treating the others as constants.

Example 4. Find all of the first-order partial derivatives for the function $w = xe^y + ye^z$.

$$w_x = (xe^y + ye^z)'_x = e^y + 0 = e^y$$

$$w_y = (xe^y + ye^z)'_y = xe^y + e^z$$

$$w_z = (xe^y + ye^z)'_z = 0 + ye^z = ye^z$$