Section 8.3. **Extrema of Functions of Two Variables**

We say that $f(a, b)$ **is a local maximum** if there exists a circular region in the domain of f with (a, b) as the center, such that

$$f(a, b) \geq f(x, y)$$

for all (x, y) in the region.

Similarly, we say that $f(a, b)$ **is a local minimum** if there exists a circular region in the domain of f with (a, b) as the center, such that

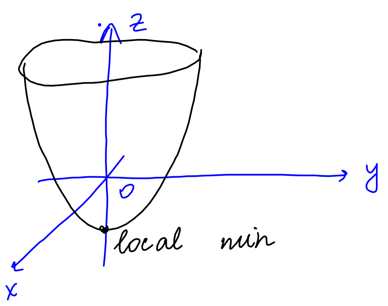
$$f(a, b) \leq f(x, y)$$

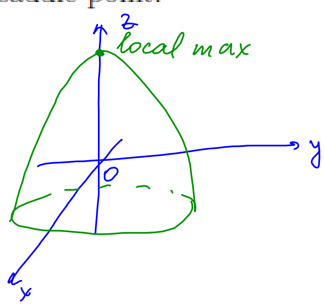
for all (x, y) in the region.

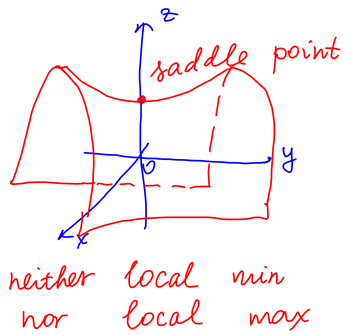
Theorem. Let $f(a, b)$ be a local extremum for the function f. If both f_x and f_y exist at (a, b), then

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0$$

Theorem gives us *necessary* (but not *sufficient*) conditions for $f(a, b)$ to be a local extremum. We thus find all points (a, b) such that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ and test further to determine whether $f(a, b)$ is a local extremum or a saddle point.







Example 1. Determine the critical point(s) of $f(x,y) = 2x^2 + 3y^2 + 2xy + 4x - 8y + 3$.

$f_x = 4x + 2y + 4 = 0$

$f_y = 6y + 2x - 8 = 0$

$$\begin{cases} \frac{4x+2y+4}{2} = \frac{0}{2} \\ \frac{6y+2x-8}{2} = \frac{0}{2} \end{cases}$$

$$\begin{cases} 2x+y+2=0 \\ 3y+x-4=0 \end{cases} \Rightarrow$$

$y = -2-2x$

$3(-2-2x)+x-4=0$

$-6-6x+x-4=0$

$-5x-10=0$

$5x=-10$

$\boxed{x=-2}$

$y = -2-2x$

$= -2-2(-2)$

$= -2+4 = \boxed{2=y}$

$\boxed{(-2,2) \text{ critical point}}$

Second-derivative test for local extrema. For $z = f(x, y)$ if

1. $f_x(a, b) = 0$ and $f_y(a, b) = 0$
2. All second-order partial derivatives of f exist in some circular region containing (a, b) as center.
3. $A = f_{xx}(a, b)$, $B = f_{xy}(a, b)$, $C = f_{yy}(a, b)$

Then

- If $AC - B^2 > 0$ and $A < 0$, then $f(a, b)$ is a local maximum.
- If $AC - B^2 > 0$ and $A > 0$, then $f(a, b)$ is a local minimum.
- If $AC - B^2 < 0$, then f has a saddle point at (a, b).
- If $AC - B^2 = 0$, the test fails.

Example 2. Find local extrema for the function

$$f(x, y) = 2x^2 - xy + y^2 - x - 5y + 8$$

$f_x = 4x - y - 1 = 0$

$f_y = -x + 2y - 5 = 0$

$\begin{cases} 4x - y - 1 = 0 \\ -x + 2y - 5 = 0 \end{cases} \Rightarrow y = 4x - 1$

$-x + 2(4x - 1) - 5 = 0$

$-x + 8x - 2 - 5 = 0$

$7x - 7 = 0$

$\boxed{x = 1}$ $\boxed{y = 4(1) - 1 = 3}$

$\boxed{(1,3) \text{ critical point}}$

$f_{xx} = \boxed{4 = A > 0}$

$f_{xy} = (4x - y - 1)'_y = \boxed{-1 = B}$

$f_{yy} = \boxed{2 = C}$

$AC - B^2 = 4(2) - (-1)^2$

$= 8 - 1 = \boxed{7 > 0}$

$\boxed{(1,3) \text{ is the local min}}$