

**Review Before Test 2.**

Test 2 will be held on Thursday, March 25. It will cover sections 3.7, 4.1–4.4, 4.7, 5.1, 5.2, 5.4–5.6.

Section 3-7. **Marginal analysis in business and economics.**

**Definition.** If  $x$  is the number of units of a product produced in some time interval, then

$$\begin{aligned} \text{total cost} &= C(x) \\ \text{marginal cost} &= C'(x) \\ \text{total revenue} &= R(x) \\ \text{marginal revenue} &= R'(x) \\ \text{total profit} &= P(x) = R(x) - C(x) \\ \text{marginal profit} &= P'(x) = R'(x) - C'(x) \end{aligned}$$

**Marginal cost and exact cost.** If  $C(x)$  is the total cost producing  $x$  items, the marginal cost function approximates the exact cost of producing the  $(x + 1)$ st item:

$$C'(x) \approx C(x + 1) - C(x)$$

Similar statement can be made for  $R(x)$  and  $P(x)$ .

**Marginal average cost, revenue, and profit.** If  $x$  is the number of units of a product produced in some time interval, then

$$\begin{aligned} \text{average cost } \bar{C}(x) &= \frac{C(x)}{x} \\ \text{marginal average cost} &= \bar{C}'(x) = (\bar{C}(x))' \\ \text{average revenue } \bar{R}(x) &= \frac{R(x)}{x} \\ \text{marginal average revenue} &= (\bar{R})'(x) = \bar{R}(x))' \\ \text{average profit } \bar{P}(x) &= \frac{P(x)}{x} \\ \text{marginal average profit} &= \bar{P}'(x) = (\bar{P}(x))' \end{aligned}$$

**Example 1.** Let

$$p = 25 - 0.01x \quad \text{and} \quad C(x) = 2x + 9000$$

where  $0 \leq x \leq 2500$ , be the price-demand equation and cost function, respectively, for the manufacture of umbrellas.

1. Find the exact cost of producing the 31st umbrella. Use the marginal cost to approximate the cost of producing the 31st umbrella.
2. Find the marginal revenue and the marginal average revenue functions.
3. Find the average profit per umbrella if 20 umbrellas is produced. Find the marginal average profit at a production level of 20 umbrellas. Estimate the average profit per umbrella if 21 umbrella is produced.

## Section 4-1 The constant $e$ and continuous compound interest.

### Continuous compound interest

$$A = Pe^{rt}$$

where

$P$  = principal

$r$  = annual nominal interest rate compounded continuously

$t$  = time in years

$A$  = amount at time  $t$ .

**Example 2.** A bank offers a 10-year certificate of deposit (CD) that earns 4.15% compounded continuously.

1. If \$10000 is invested in this CD, how much will it be worth in 10 years?
2. How long will it take for the account to be worth \$18000?

**Example 3.** A note will pay \$25000 at maturity 10 years from now. How much should you willing to pay for the note now if money is worth 5% compounded continuously?

**Example 4.** what nominal rate compounded continuously must money be invested to double in 8 years?

**Example 5.** How long will it take for the U.S. population to double if it is continues to grow at a rate of 0.975% per year?

## Section 4-2. Derivatives of exponential and logarithmic functions.

$$(e^x)' = e^x \quad (b^x)' = b^x \ln b$$
$$(\ln x)' = \frac{1}{x} \quad (\log_b x)' = \frac{1}{x \ln b}$$

## Section 4-3. Derivatives of products and quotients.

**Product rule.**  $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$ .

**Quotient rule.**  $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$ .

## Section 4-4. The chain rule.

**Chain rule.** If  $m(x) = f(g(x))$ , then

$$m'(x) = f'(g(x))g'(x).$$

### General derivative rules.

$$([u(x)]^n)' = n[u(x)]^{n-1}u'(x)$$

$$(\ln[u(x)])' = \frac{u'(x)}{u(x)}$$

$$(e^{u(x)})' = e^{u(x)}u'(x)$$

**Example 6.** Find the equation of the tangent line to the graph of the function  $f(x) = \ln(1 - x^2 + 2x^4)$  at the point where  $x = 1$ .

**Example 7.** Find the value(s) of  $x$  where the tangent line to the graph of the function  $y = 5e^{x^2-4x+1}$  is horizontal.

**Example 8.** Find each derivative

1.  $\frac{d}{dx} \log_3(\sqrt[4]{4x^3 + 5x + 7})$
2.  $\frac{d}{dx} 8^{1-2x^3}$
3.  $\frac{d}{dx} \frac{3x^2}{(x^2 + 5)^3}$
4.  $\frac{d}{dx} [(x^2 + x - 3)e^{2x+3}]$

### Section 4-7 **Elasticity of demand.**

**Definition.** If price and demand are related by  $x = f(p)$ , then the elasticity of demand is given by

$$E(p) = \frac{\text{relative rate of change of demand}}{\text{relative rate of change of price}} = -\frac{pf'(p)}{f(p)}.$$

If  $0 < E(p) < 1$ , then the demand is **inelastic**, that is the demand is not sensitive to changes in price. A change in price produces a smaller change in demand.

If  $E(p) > 1$ , then the demand is **elastic**, that is the demand is sensitive to changes in price. A change in price produces a larger change in demand.

If  $E(p) = 1$ , then the demand is **unit**. A change in price produces the same change in demand.

#### **Revenue and elasticity of demand.**

Demand is inelastic:

A price increase will increase revenue.

A price decrease will decrease revenue.

Demand is elastic:

A price increase will decrease revenue.

A price decrease will increase revenue.

**Example 9.** Given the price-demand equation

$$0.02x + p = 60$$

1. Find the elasticity of demand  $E(p)$ .
2. For which values of  $p$  is demand elastic?
3. If  $p = \$10$  and the price is increased by 5%, what is the approximate change in demand?
4. If  $p = \$40$  and the price is decreased, will revenue increase or decrease?

### Section 5-1. **First derivative and graphs.**

**Definition.** The values of  $x$  in the domain of  $f$  where  $f'(x) = 0$  or where  $f'(x)$  does not exist are called the **critical values** of  $f$ .

#### **Increasing / decreasing test**

- (a) If  $f'(x) > 0$  on an interval, then  $f$  is increasing on that interval
- (b) If  $f'(x) < 0$  on an interval, then  $f$  is decreasing on that interval

**To find intervals on which a function is increasing or decreasing**, we will construct a sign chart for  $f'(x)$  to determine which values of  $x$  make  $f'(x) > 0$  and which values make  $f'(x) < 0$ .

**Definition.** A function  $f$  has a **local maximum** (or **relative maximum**) at  $c$  if  $f(c) \geq f(x)$  when  $x$  is near  $c$ . [This means that  $f(c) \geq f(x)$  for all  $x$  in some *open* interval containing  $c$ ]. Similarly,  $f$  has a **local minimum** at  $c$  if  $f(c) \leq f(x)$  when  $x$  is near  $c$ . The quantity  $f(c)$  is called a **local extremum** if it either a local maximum or a local minimum.

**The first derivative test** Suppose that  $c$  is a critical number of a continuous function  $f$ .

(a) If  $f'$  changes from positive to negative at  $c$ , then  $f$  has a local max at  $c$ .

(b) If  $f'$  changes from negative to positive at  $c$ , then  $f$  has a local min at  $c$ .

(c) If  $f'$  does not change sign at  $c$ , then  $f$  has a no local max or min at  $c$ .

**The second derivative test** Suppose  $f''$  is continuous near  $c$ .

(a) If  $f'(c) = 0$  and  $f''(c) > 0$ , then  $f$  has a local min at  $c$ .

(b) If  $f'(c) = 0$  and  $f''(c) < 0$ , then  $f$  has a local max at  $c$ .

**Example 10.** Given the function  $f(x) = \frac{1}{4}x^4 - 4x$ .

1. Find critical values of  $f(x)$ .
2. Find intervals on which  $f(x)$  is increasing and decreasing.
3. Find local extrema for  $f(x)$ .

## Section 5-2. Second derivative and graphs.

**Definition.** The graph of the function  $f$  is **concave upward (CU)** on the interval  $(a, b)$  if  $f'(x)$  is increasing on  $(a, b)$  and is **concave downward (CD)** on the interval  $(a, b)$  if  $f'(x)$  is decreasing on  $(a, b)$

**Definition.** For  $y = f(x)$ , the **second derivative** of  $f$ , provided that it exists, is

$$f''(x) = \frac{d}{dx}f'(x)$$

**Example 11.** Find  $f''(x)$  for the functions

1.  $f(x) = x^2(2x^3 - 5)^4$
2.  $f(x) = \frac{2}{x} - \frac{6}{x^3}$

**Definition.** An **inflection point** is a point on the graph of the function where the concavity changes.

**Concavity test**

(a) If  $f''(x) > 0$  on an interval, then  $f$  is CU on this interval.

(b) If  $f''(x) < 0$  on an interval, then  $f$  is CD on this interval.

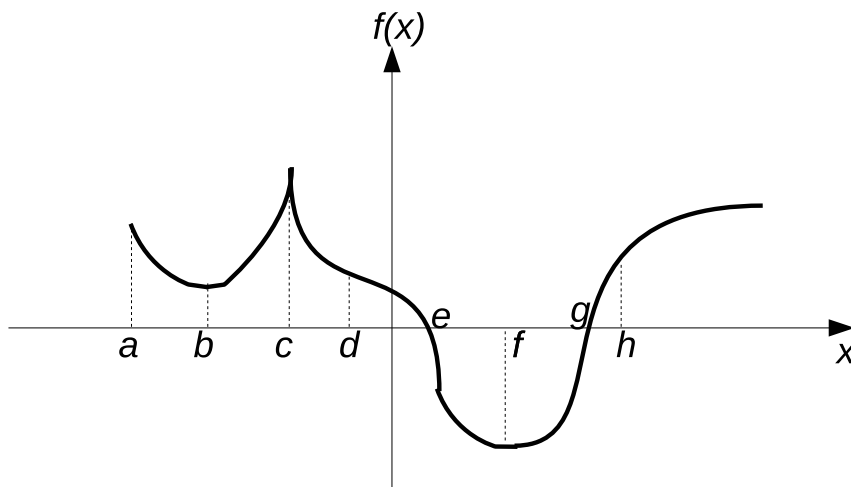
If  $y = f(x)$  is continuous on  $(a, b)$  and has an inflection point at  $x = c$ , then either  $f''(c) = 0$  or  $f''(c)$  does not exist.

**Example 11.** Given the function  $f(x) = \frac{1}{4}x^4 - 4x$ .

1. Find intervals on which  $f(x)$  is concave upward and concave downward.

2. Find all inflection points of  $f(x)$ .

**Example 12.** Given the graph of the function  $y = f(x)$ .



1. Find the intervals on which  $f'(x) > 0$ .
2. Find the intervals on which  $f'(x) < 0$ .
3. Find  $x$ -coordinates of the points where  $f'(x) = 0$ .
4. Find the intervals on which  $f''(x) > 0$ .
5. Find the intervals on which  $f''(x) < 0$ .
6. Find  $x$ -coordinates of the points where  $f''(x) = 0$ .

### Section 5-4 Curve-sketching techniques.

#### Graphing strategy.

1. Analyze  $f(x)$ .
  - (a) Find the domain of  $f$ .
  - (b) Find the intercepts.
  - (c) Find asymptotes.
2. Analyze  $f'(x)$ .
  - (a) Find all critical value of  $f(x)$ .
  - (b) Construct a sign chart for  $f'(x)$ .
  - (c) Determine the intervals on which  $f$  is increasing and decreasing.
  - (d) Find local maxima and minima.

3. Analyze  $f''(x)$ .
  - (a) Find all partition numbers of  $f''(x)$ .
  - (b) Construct a sign chart for  $f''(x)$ .
  - (c) Determine the intervals on which  $f$  is concave upward and concave downward.
  - (d) Find inflection points.
4. Sketch the graph of  $f$ .

### Section 5-5 Absolute maxima and minima.

**Definition.** If  $f(c) \geq f(x)$  for all  $x$  in the domain of  $f$ , then  $f(c)$  is called the **absolute maximum value** of  $f$ .

If  $f(c) \leq f(x)$  for all  $x$  in the domain of  $f$ , then  $f(c)$  is called the **absolute minimum value** of  $f$ .

The absolute min and the absolute max are referred to as **absolute extrema**.

**Extreme value theorem.** A function  $f$  is continuous on a closed interval  $[a, b]$  has both an absolute maximum value and an absolute minimum value on that interval.

**Locating absolute extrema.** Absolute extrema (if they exist) must always occur at critical values or at endpoints.

**Finding absolute extrema on a closed interval.**

- Step 1. Check to make certain that  $f$  is continuous over  $[a, b]$ .
- Step 2. Find the critical values in the interval  $(a, b)$ .
- Step 3. Evaluate  $f$  at the endpoints  $a$  and  $b$  and at the critical values found in Step 2.
- Step 4. The absolute maximum  $f(x)$  on  $[a, b]$  is the largest value found in Step 3.
- Step 5. The absolute minimum  $f(x)$  on  $[a, b]$  is the smallest value found in Step 3.

**Example 13.** Find the absolute maximum and absolute minimum for the function  $f(x) = \sqrt{9 - x^2}$  on the interval  $[-1, 2]$ .

**The second derivative test** Suppose  $f''$  is continuous near  $c$ .

(a) If  $f'(c) = 0$  and  $f''(c) > 0$ , then  $f$  has an absolute min at  $c$ .

(b) If  $f'(c) = 0$  and  $f''(c) < 0$ , then  $f$  has an absolute max at  $c$ .

**Note.** The second derivative test cannot be applied when  $f''(c) = 0$  or  $f''(c)$  does not exist.

**Example 14.** Find the absolute maximum and minimum for the function  $f(x) = \frac{x^2 - 1}{x^2 + 1}$ .

### Section 5-6 Optimization.

Optimization problems are problems that involve finding the absolute maximum value or the absolute minimum value of a function.

**Steps in solving optimization problems**

- Step 1. Understand the problem.

Step 2. Draw a diagram.

Step 3. Introduce variables, look for relationships among the variables, and construct a mathematical model of the form

Maximize (or minimize)  $f(x)$  on the interval  $I$

Step 4. Find the absolute maximum or minimum of  $f(x)$  on the interval  $I$  and the value(s) of  $x$  where this occurs.

Step 5. Use the solution to the mathematical model to answer all questions asked in the problem.

**Example 15.** A box with a square base and open top must have a volume of  $32000 \text{ cm}^3$ . Find the dimensions of the box that minimize the amount of material used.