MATH 304–505 Spring 2009

Sample problems for the final exam

Any problem may be altered or replaced by a different one!

1. Find a quadratic polynomial $p(x)$ such that $p(-1) = p(3) = 6$ and $p'(2) = p(1)$.

2. Let
$$A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{pmatrix}.$$

   (a) Evaluate the determinant of the matrix A.

   (b) Find the inverse matrix A^{-1}.

3. Let $\mathbf{v}_1 = (1,1,1)$, $\mathbf{v}_2 = (1,0,1)$, $\mathbf{v}_3 = (0,1,1)$, and let $\mathbf{u}_1 = (1,1,0)$, $\mathbf{u}_2 = (1,0,2)$, $\mathbf{u}_3 = (2,1,1)$. Find the transition matrix corresponding to the change of basis from $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ to $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$.

4. Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by
$$L(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{v}_1)\mathbf{v}_2, \quad \text{where} \;\; \mathbf{v}_1 = (1,2,-1), \; \mathbf{v}_2 = (2,2,1).$$

   (a) Find the matrix of the operator L.

   (b) Find bases for the range and the kernel of L.

5. Let V be a subspace of $\mathbb{R}^4$ spanned by vectors $\mathbf{x}_1 = (1,1,0,0)$, $\mathbf{x}_2 = (2,0,-1,1)$, and $\mathbf{x}_3 = (0,1,1,0)$.

   (a) Find the distance from the point $\mathbf{y} = (0,0,0,4)$ to the subspace V.

   (b) Find the distance from the point $\mathbf{y}$ to the orthogonal complement $V^{\perp}$.

6. Let $B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 3 \end{pmatrix}$.

(a) Find all eigenvalues of the matrix B.

(b) Find a basis for $\mathbb{R}^3$ consisting of eigenvectors of B.

(c) Find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of B.

(d) Find a diagonal matrix D and an invertible matrix X such that $B = XDX^{-1}$.

7. Solve a system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = 3x + y, \\ \dfrac{dy}{dt} = x + 3y \end{cases}$$

subject to the initial conditions $x(0) = y(0) = 1$.