Review for Test 1.

Vectors. The dot product.

Let c be a scalar and $\vec{a} = < a_1, a_2 > = a_1\vec{\imath} + a_2\vec{\jmath}$ and $\vec{b} = < b_1, b_2 > = b_1\vec{\imath} + b_2\vec{\jmath}$ be vectors. Then

- $c\vec{a} = < ca_1, ca_2 >$
- $\vec{a} + \vec{b} = < a_1 + b_1, a_2 + b_2 >$,
- $\vec{a} - \vec{b} = < a_1 - b_1, a_2 - b_2 >$,

the **magnitude** of $\vec{a}$ is

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2}$$

vector $\vec{u} = \frac{1}{|\vec{a}|}\vec{a} = \left\langle \frac{a_1}{|\vec{a}|}, \frac{a_2}{|\vec{a}|} \right\rangle$ is a unit vector that has the same direction as $\vec{a}$

Given the points $A(x_1, y_1)$ and $B(x_2, y_2)$, then

$$\overrightarrow{AB} = < x_2 - x_1, y_2 - y_1 >$$

the **dot product** of vectors $\vec{a}$ and $\vec{b}$:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta = a_1b_1 + a_2b_2$$

the **scalar projection** of $\vec{b}$ onto $\vec{a}$ is a number

$$\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

the **vector projection** of $\vec{b}$ onto $\vec{a}$ is a vector

$$\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2}\vec{a}$$

Definition. Given the nonzero vector $\vec{a} = < a_1, a_2 >$, the **orthogonal complement** of $\vec{a}$ is the vector

$$\vec{a}^{\perp} = < -a_2, a_1 >$$

Example 1. Given vectors $\vec{a} = \vec{\imath} - 2\vec{\jmath}$, $\vec{b} = < -2, 3 >$. Find

(a) a unit vector $\vec{u}$ that has the same direction as $2\vec{b} + \vec{a}$.

$$2\vec{b} + \vec{a} = 2 < -2, 3 > + < 1, -2 > = < -4, 6 > + < 1, -2 > = < -3, 4 >$$

$$\vec{u} = \frac{2\vec{b} + \vec{a}}{|2\vec{b} + \vec{a}|} = \frac{< -3, 4 >}{\sqrt{9 + 16}} = \frac{1}{5} < -3, 4 > = < -\frac{3}{5}, \frac{4}{5} >$$

(b) $\text{comp}_{\vec{b}}\vec{a}$, $\text{proj}_{\vec{b}}\vec{a}$.

$$\text{comp}_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{(1)(-2)+(-2)(3)}{\sqrt{4+9}} = -\frac{8}{\sqrt{13}}$$

$$\text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b} = -\frac{8}{13}< -2, 3 > = < \frac{16}{13}, \frac{-24}{13} >$$

The **distance** D from a point (x_0, y_0) to a line $ax + by = c$ is

$$D = \left| \frac{ax_0 + by_0 - c}{\sqrt{a^2 + b^2}} \right|$$

Example 2. Find the distance from the point (1,2) to the line $x + 3y = -3$.

$$D = \left| \frac{1 + (3)(2) - (-3)}{\sqrt{1+9}} \right| = \left| \frac{1+6+3}{\sqrt{10}} \right| = \frac{10}{\sqrt{10}} = \sqrt{10}$$

Vector equation of a line

A line L is determined by a point P_0 on L and a direction. Let $\vec{v}$ be a vector parallel to line L. Let P be be an arbitrary point on L and let $\vec{r_0}$ and $\vec{r}$ be the position vectors of P and P_0. Then the **vector equation** of line L is

$\vec{r}(t) = \vec{r_0} + t\vec{v}$.

If $\vec{r} =< x(t), y(t) >$, $\vec{v} =< a, b >$ and $P(x_0, y_0)$ then **parametric equations** of the line L are

$x(t) = x_0 + at$, $y(t) = y_0 + bt$.

Example 3. Find the vector, parametric, and the Cartesian equations for the line passing through the points $A(1, -3)$ and $B(2, 1)$.

$\vec{AB} = < 1, 4 >$

the line is parallel to $\vec{AB}$ and passes through A.

vector equation: $< x, y > = < 1, -3 > + t < 1, 4 >$

parametric equations: $\begin{cases} x = 1 + t \\ y = -3 + 4t \end{cases}$

Cartesian equation: $t = x - 1$

$y = -3 + 4(x-1)$

$y = -3 + 4x - 4$

$\boxed{y = 4x - 7}$

Example 4. Evaluate each of the following limits

(a.) $\lim_{x\to 5} \frac{x^2-5x+10}{x^2-25} = \lim_{x\to 5} \frac{5^2-(5)(5)+10}{5^2-25} = \frac{\cancel{0}10}{0} = \infty$

(b.) $\lim_{x\to 7} \frac{2-\sqrt{x-3}}{x^2-49} = \lim_{x\to 7} \frac{(2-\sqrt{x-3})(2+\sqrt{x-3})}{(x^2-49)(2+\sqrt{x-3})} = \lim_{x\to 7} \frac{4-(x-3)}{(x^2-49)(2+\sqrt{x-3})}$

$= \lim_{x\to 7} \frac{7-x}{(x+7)(x-7)(2+\sqrt{x-3})} = \lim_{x\to 7} \frac{-1}{(x+7)(2+\sqrt{x-3})}$

$= -\frac{1}{56}$

(c.) $\lim_{t\to 1} \left\langle \frac{t^2-2t+1}{t-1}, \frac{\sqrt{t}-1}{t^2-1} \right\rangle = \left\langle \lim_{t\to 1} \frac{t^2-2t+1}{t-1}, \lim_{t\to 1} \frac{\sqrt{t}-1}{t^2-1} \right\rangle$

$= \left\langle \lim_{t\to 1} \frac{(t-1)^2}{t-1}, \lim_{t\to 1} \frac{\sqrt{t}-1}{(t-1)(t+1)} \right\rangle$

$= \left\langle \lim_{t\to 1} (t-1), \lim_{t\to 1} \frac{\sqrt{t}-1}{(\sqrt{t}-1)(\sqrt{t}+1)(t+1)} \right\rangle$

$= \left\langle 0, \lim_{t\to 1} \frac{1}{(\sqrt{t}+1)(t+1)} \right\rangle = \left\langle 0, \frac{1}{4} \right\rangle$

(d.) $\lim_{y\to\infty} \frac{7y^3+4y}{2y^3-y^2+3}$

$= \lim_{y\to\infty} \frac{y^3\left(7+\frac{4}{y^2}\right)}{y^3\left(2-\frac{y^2}{y^3}+\frac{3}{y^3}\right)} = \frac{7}{2}$

(e.) $\lim_{x\to\infty}(\sqrt{x^2+3x+1}-x) = \lim_{x\to\infty} \frac{(\sqrt{x^2+3x+1}-x)(\sqrt{x^2+3x+1}+x)}{\sqrt{x^2+3x+1}+x}$

$= \lim_{x\to\infty} \frac{x^2+3x+1-x^2}{\sqrt{x^2+3x+1}+x} = \lim_{x\to\infty} \frac{3x+1}{\sqrt{x^2(1+\frac{3}{x}+\frac{1}{x^2})}+x}$

$= \lim_{x\to\infty} \frac{3x+1}{\sqrt{x^2}+x} = \lim_{x\to\infty} \frac{3x+1}{2x} = \frac{3}{2}$

The Squeeze Theorem. If $f(x) \le g(x) \le h(x)$ for all x in an open interval that contains a (except possibly at a) and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\boxed{\lim_{x\to a} g(x) = L}$

Continuity. Vertical and horizontal asymptotes
Function f is **continuous at a number** a if $\lim_{x\to a} f(x) = f(a)$.

If f is not continuous at a, then f has **discontinuity** at a:

- if $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$, then f has a **jump discontinuity** at a,
- if either $\lim_{x\to a^+} f(x) = \infty$ or $\lim_{x\to a^-} f(x) = \infty$, then f has an **infinity discontinuity** at a and we say line $x = a$ is a **vertical asymptote** of the curve $y = f(x)$.
- if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) \neq f(a)$, then f has a **removable discontinuity** at a.

If $\lim_{x\to\infty} f(x) = b$, then we say line $y = b$ is **horizontal asymptote** of the curve $y = f(x)$.

Example 5.

(a.) Find and classify all points of discontinuity for the function

$$f(x) = \begin{cases} x^2+1 & , \quad \text{if } x < 2, \\ x+2 & , \quad \text{if } x \ge 2. \end{cases}$$

$f(x)$ is continuous if $x<2$ and $x>2$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2}(x^2+1) = 5$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2}(x+2) = 4$

$\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$

f has the jump discontinuity at 2.

(b) Find the vertical and horizontal asymptotes of the curve $y = \frac{x^2+4}{3x^2-3}$.

vertical: $3x^2 - 3 = 0$

$x^2 - 1 = 0$

$\boxed{x = 1,\ x = -1 \text{ — vertical asymptotes}}$

$\lim_{x \to \infty} \frac{x^2+4}{3x^2-3} = \frac{1}{3}$

$\boxed{y = \frac{1}{3} \text{ — horizontal asymptote.}}$

The intermediate value theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number strictly between $f(a)$ and $f(b)$. Then there exist a number c in (a, b) such that $f(c) = N$.

Example 6. Use the intermediate value theorem to show that there is a root of the equation $x^3 - 3x + 1 = 0$ in the interval (1,2).

Let $f(x) = x^3 - 3x + 1$

$f(1) = 1 - 3 + 1 = -1 < 0$

$f(2) = 8 - 6 + 1 = 3 > 0$

Since $f(1) < 0$ and $f(2) > 0$ and f is continuous on $(1,2)$, then, by IVT, there is a ~~point~~ number $1 < c < 2$ such that $f(c) = 0$.

$x = c$ is a root to the equation $x^3 - 3x + 1 = 0$

Derivatives

Definition. The **derivative of a function** f **at a number** a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

if the limit exist.

Example 7. Find $f'(9)$ if $f(x) = \sqrt{x}$ by using the definition of derivative.

$$f'(9) = \lim_{x \to 9} \frac{f(x) - f(9)}{x - 9}$$

$$= \lim_{x \to 9} \frac{\sqrt{x} - \sqrt{9}}{x - 9}$$

$$= \lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$$

$$= \lim_{x \to 9} \frac{\sqrt{x} - 3}{(\sqrt{x} - 3)(\sqrt{x} + 3)}$$

$$= \lim_{x \to 9} \frac{1}{\sqrt{x} + 3}$$

$$= \boxed{\frac{1}{6}}$$

Tangent line

If $f'(a)$ exist, then the equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$ is

$$y - f(a) = f'(a)(x - a)$$

Example 8. Find the equation of the tangent line to the curve $y = \sqrt{5 - x}$ at the point (1,2).

$f(x) = \sqrt{5-x}$

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \to 1} \frac{\sqrt{5-x} - 2}{x - 1} = \lim_{x \to 1} \frac{(\sqrt{5-x} - 2)(\sqrt{5-x} + 2)}{(x-1)(\sqrt{5-x} + 2)}$$

$$= \lim_{x \to 1} \frac{5 - x - 4}{(x-1)(\sqrt{5-x} + 2)}$$

$$= \lim_{x \to 1} \frac{1 - x}{(x-1)(\sqrt{5-x} + 2)}$$

$$= \lim_{x \to 1} \frac{-1}{\sqrt{5-x} + 2}$$

$$= -\frac{1}{4}$$

tangent line:

$$\boxed{y = -\frac{1}{4}(x-1) + 2}$$

Let $\vec{r}(t) =< x(t), y(t) >$ be a vector function. Then the equation of the tangent line to a curve traced by $\vec{r}(t)$ at the point P corresponding to the vector $\vec{r}(a) =< x(a), y(a) >$ is given by

$$\vec{L}(t) = \vec{r}(a) + t\vec{v}$$

where

$$\vec{v} = \vec{r}'(a) =< x'(a), y'(a) >$$

$$x'(a) = \lim_{h\to 0} \frac{x(a+h) - x(a)}{h}, \quad y'(a) = \lim_{h\to 0} \frac{y(a+h)-y(a)}{h}$$

Example 9. Find the tangent vector and parametric equations for the line tangent to the curve $\vec{r}(t) =< t^2 + 2t, t^3 - t >$ at the point corresponding to $t = 1$.

$\vec{r}(1) = < 3, 0 >$

$\vec{v}(1) = <4,2>$ tangent vector $\vec{v}'(t)$:

$$(t^2+2t)'_{t=1} = \lim_{h\to 0} \frac{(1+h)^2 + 2(1+h) - 3}{h}$$

$$= \lim_{h\to 0} \frac{1+2h+h^2+2+2h-3}{h}$$

$$= \lim_{h\to 0} \frac{4h+h^2}{h} = \lim_{h\to 0} \frac{h(4+h)}{h} = 4$$

$$(t^3-t)'_{t=1} = \lim_{h\to 0} \frac{(1+h)^3 - (1+h) - 0}{h} = \lim_{h\to 0} \frac{(1+h)^3-(1+h)}{h}$$

$$= \lim_{h\to 0} \frac{(1+h)[(1+h)^2-1]}{h} = \lim_{h\to 0} \frac{(1+h)[1+2h+h^2-1]}{h}$$

$$= \lim_{h\to 0} \frac{(1+h)(2h+h^2)}{h} = \lim_{h\to 0} \frac{(1+h)(h)(2+h)}{h} = 2$$

tangent line:

$<x,y> =$

$<3,0>$

$+t<4,2>$

vector equation.

Parametric equations:

$x = 3+4t$

$y = 2t$

Velocity

Suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement of the object from the origin at time t. Function f is called the **position function** of the object. Then the **velocity** or **instantaneous velocity** at time t is

$$v(t) = s'(t)$$

Example 10. The displacement of an object moving in a straight line is given by $s(t) = 1 + 2t + t^2/4$ (t is in seconds). Find the velocity of the object when $t = 1$.

$$\vec{v}(1) = \lim_{t\to 1} \frac{s(t)-s(1)}{t-1} = \lim_{t\to 1} \frac{1+2t+\frac{t^2}{4}-1-2-\frac{1}{4}}{t-1}$$

$$= \lim_{t\to 1} \frac{2t+\frac{t^2}{4}-\frac{9}{4}}{t-1} = \lim_{t\to 1} \frac{(t-1)\left(\frac{t}{4}+\frac{9}{4}\right)}{t-1}$$

$$= \lim_{t\to 1} \left(\frac{t}{4}+\frac{9}{4}\right)$$

$$= \frac{10}{4}$$

$$= \boxed{\frac{5}{2}}$$

Suppose an object moves in the xy-plane in such a way that its position at time t is given by the position vector $\vec{r}(t) =< x(t), y(t) >$. The velocity $\vec{v}(t)$ at the time t is

$$\vec{v}(t) = \vec{r}'(t) =< x'(t), y'(t) >$$

The **speed** of a particle is defined to be the magnitude of the velocity vector.

Example 11. The vector function $\vec{r}(t) = (t^2 - 4t)\vec{\imath} + (2t+1)\vec{\jmath}$ represents the position of a particle at time t.

(a) Find the velocity of the particle when $t = 1$

$\vec{r}(1) = \langle (1-4), 2+1 \rangle = \langle -3, 3 \rangle.$

$x(t) = t^2 - 4t, \quad y(t) = 2t+1$

$$x'(1) = \lim_{t\to 1} \frac{x(t) - x(1)}{t-1} = \lim_{t\to 1} \frac{t^2-4t+3}{t-1}$$

$$= \lim_{t\to 1} \frac{(t-1)(t-3)}{t-1} = \lim_{t\to 1} (t-3) = -2$$

$$y'(1) = \lim_{t\to 1} \frac{y(t)-y(1)}{t-1} = \lim_{t\to 1} \frac{2t+1-3}{t-1}$$

$$= \lim_{t\to 1} \frac{2t-2}{t-1} = \lim_{t\to 1} \frac{2(t-1)}{t-1} = 2$$

$\vec{v}(1) = \langle -2, 2 \rangle$

(b) Find the speed of the particle when $t = 1$

$\text{speed} = |\vec{v}(1)| = \sqrt{4+4} = \sqrt{8} = \boxed{2\sqrt{2}}$