

Review for Test 2.

Section 3.2 Differentiation formulas.

Table of derivatives

1. $(C)' = 0$, C is a constant
2. $(x)' = 1$,
3. $(x^2)' = 2x$,
4. $(x^n)' = nx^{n-1}$

Differentiation formulas

Suppose c is a constant and both functions $f(x)$ and $g(x)$ are differentiable.

- (a) $(cf(x))' = cf'(x)$,
- (b) $(f(x) + g(x))' = f'(x) + g'(x)$,
- (c) $(f(x) - g(x))' = f'(x) - g'(x)$,
- (d) $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$,
- (e) $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$.

Example 1. Find the derivative $f'(x)$ of each function $f(x)$:

1. $f(x) = (x^5 + 3x^3 - x + 10)(x^4 - 2x^2 + 4)$

2. $f(x) = \frac{x^2 + 3x + 3}{\sqrt{x} + 2}$

3. $f(x) = x + x\sqrt[3]{x^2} + x\sqrt{x} + \frac{1}{x\sqrt{x}} + \frac{x}{\sqrt[3]{x}}$

Tangent line

If $f'(a)$ exist, then the equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$ is

$$y - f(a) = f'(a)(x - a)$$

Example 2. Find the equation of the tangent line to the curve $y = x\sqrt{5-x}$ at the point (1,2).

Velocity

Suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement of the object from the origin at time t . Function f is called the **position function** of the object. Then the **velocity** or **instantaneous velocity** at time t is

$$v(t) = s'(t)$$

Example 3. A particle moves on a vertical line so that its coordinate at time t is $y = t^3 - 12t + 3$, $t \geq 0$.

(a.) Find the velocity and acceleration functions.

(b.) When is the particle moving upward?

(c.) Find the distance that particle travels in the time interval $0 \leq t \leq 3$

Section 3.4 Derivatives of trigonometric functions

$$\lim_{x \rightarrow 0} \sin x = 0 \quad \lim_{x \rightarrow 0} \cos x = 1$$
$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \quad \lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0$$

Derivatives

$$\frac{d}{dx} \sin x = \cos x \quad \frac{d}{dx} \cos x = -\sin x$$
$$\frac{d}{dx} \tan x = \frac{1}{\cos^2 x} = \sec^2 x \quad \frac{d}{dx} \cot x = -\frac{1}{\sin^2 x} = -\csc^2 x$$
$$\frac{d}{dx} \csc x = -\csc x \cot x \quad \frac{d}{dx} \sec x = \sec x \tan x$$

Example 4.

(a) Find $\lim_{x \rightarrow 0} \frac{\sin 5x}{\tan 2x}$

(b) Find y' if $y = \sqrt{\csc 2x + \tan^2(x^2 + 1)}$

Section 3.5 The Chain Rule

If the derivatives $g'(x)$ and $f'(g(x))$ both exist, and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then $F'(x)$ exists and is given by the product

$$F'(x) = f'(g(x))g'(x)$$

Example 5. Find y' if $y = \sqrt{xe^x + x}$

Section 3.6 Implicit differentiation

Example 6. Find $\frac{dy}{dx}$ if $\sin(x + y) = y^2 \tan x$

Two curves are called **orthogonal** if at each point of intersection their tangent lines are perpendicular.

Example 7. Show that the curves $2x^2 + y^2 = 3$ and $x = y^2$ are orthogonal.

Section 3.7 Derivatives of vector functions

If $\vec{r}(t) = \langle x(t), y(t) \rangle$ is a vector function, then

$$\vec{r}'(t) = \langle x'(t), y'(t) \rangle$$

if $x'(t)$ and $y'(t)$ exist.

Let $\vec{r}(t) = \langle x(t), y(t) \rangle$ be a vector function. Then the equation of the tangent line to a curve traced by $\vec{r}(t)$ at the point P corresponding to the vector $\vec{r}(a) = \langle x(a), y(a) \rangle$ is given by

$$\vec{L}(t) = \vec{r}(a) + t\vec{v}, \text{ where } \vec{v} = \vec{r}'(a) = \langle x'(a), y'(a) \rangle$$

Example 7. Find the tangent vector and parametric equations for the line tangent to the curve $\vec{r}(t) = \langle t^2 + 2t, t^3 - t \rangle$ at the point corresponding to $t = 1$.

velocity at time $t = \vec{r}'(t) = \langle x'(t), y'(t) \rangle$

speed at time $t = |\vec{r}'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$

acceleration at time $t = \vec{r}''(t) = \langle x''(t), y''(t) \rangle$

Example 8. The vector function $\vec{r}(t) = \langle t, 25t - 5t^2 \rangle$ represents the position of a particle at time t . Find the velocity, speed, and acceleration at $t = 1$.

Section 3.8 Higher derivatives

$$f''(x) = [f'(x)]', \quad f'''(x) = [f''(x)]', \dots, f^{(n)}(x) = [f^{(n-1)}(x)]'$$

For the vector function $\vec{r}(t) = \langle x(t), y(t) \rangle$

$$\vec{r}'(t) = \langle x'(t), y'(t) \rangle, \quad \vec{r}''(t) = [\vec{r}'(t)]' = \langle x''(t), y''(t) \rangle, \dots$$

Example 9. Find y'' if $y = e^{-5x} \cos 3x$

Section 3.9 Slopes and tangents to parametric curves

Suppose that curve C is given by the parametric equation $x = x(t)$, $y = y(t)$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Example 10. At what points on the curve $x = t^3 + 4t$, $y = 6t^2$ is the tangent parallel to the line with the equations $x = -7t$, $y = 12t - 5$?

Section 3.10 **Related rates**

Strategy

1. Read the problem carefully.
2. Draw a diagram if possible.
3. Introduce notation. Assign symbols to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives.
5. Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution.
6. Use the Chain Rule to differentiate both sides of the equation with respect to t .
7. Substitute the given information into the resulting equation and solve for the unknown rate.

Example 11. A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 0.9 ft/s, how fast is the angle between the ladder and the ground changing when the bottom of the ladder is 8 ft from the wall?

Example 12. A water trough is 10 m long and has a cross-section in the shape of an isosceles trapezoid that is 40 cm wide at the bottom, 100 cm wide at the top, and has height 60 cm. If the trough is being filled with water at the rate of $0.1m^3/min$ how fast is the water level rising when the water is 40 cm deep?

Section 3.11 Differentials; linear and quadratic approximations

Definition Let $y = f(x)$, where f is a differentiable function. Then the **differential** dx is an independent variable; that is dx can be given the value of any real number. The **differential** dy is then defined in terms of dx by the equation

$$dy = f'(x)dx$$

Example 13. The circumference of a sphere was measured to be 84 cm with a possible error of 0.5 cm. Estimate the maximum error in the calculated surface area.

Suppose that $f(a)$ is a known number and the approximate value is to be calculated for $f(a + \Delta x)$ where Δx is small. Then

$$f(a + \Delta x) \approx f(a) + f'(a)\Delta x$$

Example 14. Use differentials to find an approximate value for $(1.97)^6$.

The approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the **linear approximation** or **tangent line approximation** of f at a , and the function

$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of f at a .

The **quadratic approximation** of f near a is

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$$

Example 15. Find the linear and quadratic approximation to $f(x) = \frac{1}{1+x^2}$ near 1.

Chapter 4. Inverse functions: exponential, logarithmic, and inverse trigonometric functions

Section 4.1 Exponential functions and their derivatives

An **exponential function** is a function of the form $f(x) = a^x$ where a is a positive constant. If $a > 0$ and $a \neq 1$, then $f(x) = a^x$ is continuous function with domain $(-\infty, \infty)$ and range $(0, \infty)$.

If $0 < a < 1$, $f(x) = a^x$ is decreasing function

if $a > 1$, $f(x) = a^x$ is increasing function

If $a, b > 0$ and x, y are reals, then

$$1. a^{x+y} = a^x a^y \quad 2. a^{x-y} = \frac{a^x}{a^y} \quad 3. (a^x)^y = a^{xy} \quad 4. (ab)^x = a^x b^x$$

If $0 < a < 1$, $\lim_{x \rightarrow -\infty} a^x = \infty$, $\lim_{x \rightarrow \infty} a^x = 0$

If $a > 1$, $\lim_{x \rightarrow -\infty} a^x = 0$, $\lim_{x \rightarrow \infty} a^x = \infty$
 e is the number such that

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$$

$$e \approx 2.71828182845904523536$$

$$(e^x)' = e^x \quad (e^{g(x)})' = e^{g(x)} g'(x)$$

Example 16.

(a) Find $\lim_{x \rightarrow \infty} \frac{e^{2x}}{e^{2x} + 1}$

(b) Find y' if $y = e^{x \cos x}$

Section 4.2 Inverse functions

Definition. A function f with domain A is called **one-to-one function** if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

Horizontal line test. A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Definition. Let f be one-to-one function with domain A and range B . Then its **inverse function** f^{-1} has domain B and range A and is defined by $f^{-1}(y) = x \iff f(x) = y$ for any y in B .

$$\begin{aligned} \text{domain of } f^{-1} &= \text{range of } f \\ \text{range of } f^{-1} &= \text{domain of } f \end{aligned}$$

Let f be one-to-one function with domain A and range B . If $f(a) = b$, then $f^{-1}(b) = a$.

Cancellation equations

$$f^{-1}(f(x)) = x \text{ for every } x \in A$$

$$f(f^{-1}(x)) = x \text{ for every } x \in B$$

How to find the inverse function of a one-to-one function f

1. Write $y = f(x)$
2. Solve this equation for x in terms of y .
3. Interchange x and y . The resulting equation is $y = f^{-1}(x)$.

The graph of f^{-1} is obtained by the reflecting the graph f about the line $y = x$.

Example 17. Find f^{-1} if $f(x) = \frac{1 + 3x}{5 - 2x}$.

Theorem. If f is a one-to-one differentiable function with inverse function $g = f^{-1}$ and $f'(g(a)) \neq 0$, then the inverse function is differentiable at a and

$$g'(a) = \frac{1}{f'(g(a))}$$

Example 18. Suppose g is the inverse function of f . Both functions are differentiable. Assuming that $f(1) = 3$, $f(3) = 1$, $f'(1) = 4$, $f'(3) = \frac{1}{2}$. Find $g'(3)$.

Example 19. If $f(x) = x + x^2 + e^x$ and $g(x) = f^{-1}(x)$, find $g'(1)$.