

Table of derivatives

- | | |
|--|---|
| 1. $(C)' = 0$, C is a constant, | 11. $(\tan x)' = \sec^2 x$, |
| 2. $(x)' = 1$, | 12. $(\cot x)' = -\csc^2 x$, |
| 3. $(x^2)' = 2x$, | 13. $(\sec x)' = \sec x \tan x$, |
| 4. $(x^n)' = nx^{n-1}$, | 14. $(\csc x)' = -\csc x \cot x$, |
| 5. $(\ln x)' = \frac{1}{x}$, | 15. $(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$, |
| 6. $(\log_a x)' = \frac{1}{x \ln a}$, | 16. $(\cos^{-1} x)' = -\frac{1}{\sqrt{1-x^2}}$, |
| 7. $(e^x)' = e^x$, | 17. $(\tan^{-1} x)' = \frac{1}{1+x^2}$, |
| 8. $(a^x)' = a^x \ln a$, | 18. $(\cot^{-1} x)' = -\frac{1}{1+x^2}$, |
| 9. $(\sin x)' = \cos x$, | 19. $(\sec^{-1} x)' = \frac{1}{x\sqrt{x^2-1}}$, |
| 10. $(\cos x)' = -\sin x$, | 20. $(\csc^{-1} x)' = -\frac{1}{x\sqrt{x^2-1}}$. |

Section 4.3 Logarithmic functions

$$\log_a x = y \iff a^y = x$$

The cancellation equations

$$\log_a a^x = x \quad a^{\log_a x} = x$$

Theorem. Function $f(x) = \log_a x$ is one-to-one continuous function with domain $(0, \infty)$ and range $(-\infty, \infty)$.

If $a > 1$, then

$f(x) = \log_a x$ is increasing function

$$\lim_{x \rightarrow \infty} \log_a x = \infty \quad \lim_{x \rightarrow 0^+} \log_a x = -\infty$$

if $0 < a < 1$, then

$f(x) = \log_a x$ is decreasing function

$$\lim_{x \rightarrow \infty} \log_a x = -\infty \quad \lim_{x \rightarrow 0^+} \log_a x = \infty$$

$$\log_e x = \ln x$$

If $x, y > 0$ and k is a constant, then

1. $\log_a xy = \log_a x + \log_a y$
2. $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. $\log_a x^k = k \log_a x \quad \log_a \left(\frac{1}{x}\right) = -\log_a x$.
4. $\log_{a^k} x = \frac{1}{k} \log_a x \quad \log_{\frac{1}{a}} x = -\log_a x$
5. $\log_a a = 1$

6. $\log_a 1 = 0$

7. $\log_a x = \frac{\log_b x}{\log_b a}$

Example 1. Solve the equation:

(a.) $\ln(x + 6) + \ln(x - 3) = \ln 5 + \ln 2$

(b.) Find the solution to the equation $2^x + 3 \cdot 2^x = 24$

Section 4.4 Derivatives of logarithmic functions

$$(\ln x)' = \frac{1}{x} \quad (\ln(g(x)))' = \frac{g'(x)}{g(x)}$$

Example 2. Find the derivative to the function $f(x) = \ln(\sin^{-1}(x^2))$

Logarithmic differentiation

Steps in logarithmic differentiation

1. Take the logarithm of both sides of an equation.
2. Differentiate implicitly with respect to x .
3. Solve the resulting equation for y' .

Example 3. Differentiate each function:

(a.) $f(x) = \frac{\sqrt[3]{3x-1} (x-2)^3}{2\sqrt{x+1}}$

(b.) $f(x) = (x + x^2)^{\tan x}$

Section 4.5 Exponential growth and decay

If $y(t)$ is the value of a quantity y at time t and if the rate of change of y with respect to t is proportional to $y(t)$ at any time, then

$$\frac{dy}{dt} = ky$$

where k is a constant. This equation is called the **law of natural growth** if $k > 0$ or the **law of natural decay** if $k < 0$.

The only solution to this equation is

$$y(t) = y(0)e^{kt}$$

Example 4. A bacteria culture starts with 4000 bacteria and the population triples every half-hour.

(a.) Find an expression for the number of bacteria after t hours.

(b.) Find the number of bacteria after 20 min.

(c.) When will the population reach 20,000?

Section 4.6 Inverse trigonometric functions

Inverse sine function

$$\arcsin x = \sin^{-1} x = y \Leftrightarrow \sin y = x$$

$$\text{DOMAIN} \quad -1 \leq x \leq 1$$

$$\text{RANGE} \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

CANCELLATION EQUATIONS

$$\sin^{-1}(\sin x) = x \quad \text{for } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

$$\sin(\sin^{-1} x) = x \quad \text{for } -1 \leq x \leq 1$$

Inverse cosine function

$$\arccos x = \cos^{-1} x = y \Leftrightarrow \cos y = x$$

$$\text{DOMAIN} \quad -1 \leq x \leq 1$$

$$\text{RANGE} \quad 0 \leq y \leq \pi$$

CANCELLATION EQUATIONS

$$\cos^{-1}(\cos x) = x \quad \text{for } 0 \leq x \leq \pi$$

$$\cos(\cos^{-1} x) = x \quad \text{for } -1 \leq x \leq 1$$

Inverse tangent function

$$\arctan x = \tan^{-1} x = y \Leftrightarrow \tan y = x$$

$$\text{DOMAIN} \quad -\infty \leq x \leq \infty$$

$$\text{RANGE} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

CANCELLATION EQUATIONS

$$\tan^{-1}(\tan x) = x \quad \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$\tan(\tan^{-1} x) = x \quad \text{for } -\infty < x < \infty$$

$$\lim_{x \rightarrow -\infty} \tan^{-1} x = -\frac{\pi}{2}$$

$$\lim_{x \rightarrow \infty} \tan^{-1} x = \frac{\pi}{2}$$

Example 5. Find $\cos^{-1}\left(\sin \frac{5\pi}{4}\right)$

Example 6. Derive the following formulas:

$$(a.) \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

$$(a.) \frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$

$$(a.) \frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$$

Example 7. Find the derivative of the function $f(x) = \sin^{-1}(\tan^{-1}(2x^2 + 3))$

Section 4.8 Indeterminate forms and L'Hospital's rule

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ for points close to a (except, possibly a). Suppose that

$\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = 0$ or $\lim_{x \rightarrow a} f(x) = \infty$ and $\lim_{x \rightarrow a} g(x) = \infty$. Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \left| \frac{0}{0} \text{ or } \frac{\infty}{\infty} \right| = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Indeterminate products If $\lim_{x \rightarrow a} f(x) = \infty$ and $\lim_{x \rightarrow a} g(x) = 0$ or $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = \infty$, then

$$\lim_{x \rightarrow a} f(x)g(x) = |\infty \cdot 0| = \lim_{x \rightarrow a} \frac{f(x)}{1/g(x)} = \lim_{x \rightarrow a} \frac{g(x)}{1/f(x)} = \left| \frac{0}{0} \text{ or } \frac{\infty}{\infty} \right|$$

and now we can use L'Hospital's Rule.

Indeterminate differences If we have to find $\lim_{x \rightarrow a} (f(x) - g(x)) = |\infty - \infty|$ ($\lim_{x \rightarrow a} f(x) = \infty, \lim_{x \rightarrow a} g(x) = \infty$), then we have to convert the difference into a quotient (by using a common denominator or rationalization, or factoring out a common factor) so that we have an indeterminate form of type $\left| \frac{0}{0} \text{ or } \frac{\infty}{\infty} \right|$ and we can use L'Hospital's Rule.

Indeterminate powers $\lim_{x \rightarrow a} [f(x)]^{g(x)} = |0^0 \text{ or } \infty^0 \text{ or } 1^\infty| = \lim_{x \rightarrow a} e^{g(x) \ln f(x)} = e^{\lim_{x \rightarrow a} [g(x) \ln f(x)]}$.

Now let's find

$$\lim_{x \rightarrow a} [g(x) \ln f(x)] = |0 \cdot \infty| = \lim_{x \rightarrow a} \frac{\ln f(x)}{1/g(x)} = \left| \frac{0}{0} \text{ or } \frac{\infty}{\infty} \right| = b$$

Then

$$\lim_{x \rightarrow a} [f(x)]^{g(x)} = e^b$$

Example 8. Evaluate each limit:

(a.) $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$

(b.) $\lim_{x \rightarrow \infty} (x - \sqrt{x^2 - 1})$

(c.) $\lim_{x \rightarrow 0} \left(\frac{1}{x}\right)^{\tan x}$

Section 5.1 **What does f' say about f ?**

If $f'(x) > 0$ on an interval, then f is increasing on that interval

If $f'(x) < 0$ on an interval, then f is decreasing on that interval

f has a **local maximum** at the point, where its derivative changes from positive to negative.

f has a **local minimum** at the point, where its derivative changes from negative to positive.

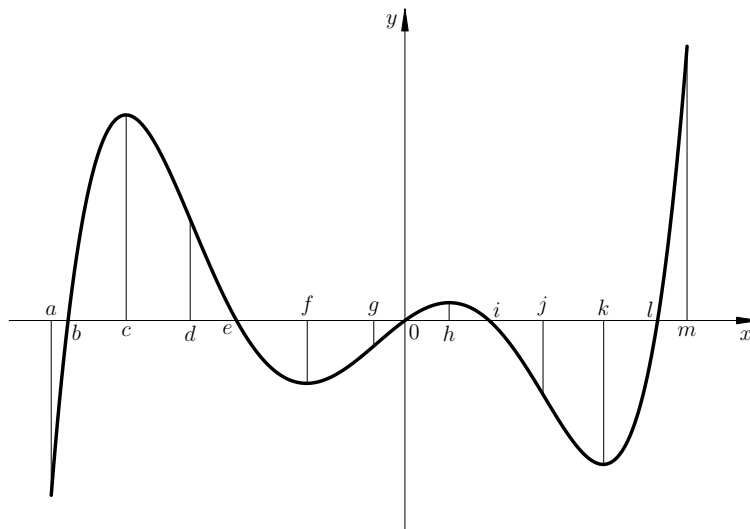
What does f'' say about f ?

If $f''(x) > 0$ on an interval, then f is **concave upward (CU)** on that interval

If $f''(x) < 0$ on an interval, then f is **concave downward (CD)** on that interval

A point where curve changes its direction of concavity is called an **inflection point**

Example 9. Given the graph of $f'(x)$.



(a.) Identify intervals on which f is increasing.

Is decreasing.

(b.) Identify the x coordinates of the points where f has local maxima.

Local minima.

(c.) Identify intervals on which f is concave upward.

Concave downward.

(d.) Find the x -coordinates of inflection points.

Section 5.2 Maximum and minimum values

Definition. A function f has an **absolute maximum** or (**global maximum**) at c if $f(c) \geq f(x)$ for all x in D , where D is the domain of f . The number $f(c)$ is called the **maximum value** of f on D . Similarly, f has an **absolute minimum** or **global minimum** at c if $f(c) \leq f(x)$ for all x in D and the number $f(c)$ is called the **minimum value** of f on D . The maximum and the minimum values of f are called the **extreme values** of f .

Definition. A function f has a **local maximum** (or **relative maximum**) at c if $f(c) \geq f(x)$ when x is near c . [This means that $f(c) \geq f(x)$ for all x in some *open* interval containing c]. Similarly, f has a **local minimum** at c if $f(c) \leq f(x)$ when x is near c .

The extreme value theorem. If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Fermat's theorem. If f has a local maximum or minimum at c , and if $f'(c)$ exists, then $f'(c) = 0$

Definition. A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

If f has a local extremum at c , then c is a critical number of f .

The closed interval method. To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

- (a) Find the values of f at the critical numbers of f in (a, b)
- (b) Find $f(a)$ and $f(b)$
- (c) The largest number of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example 10. Find the absolute maximum and absolute minimum values of $f(x) = x^3 - 2x^2 + x$ on $[-1, 1]$.

Section 5.3 Derivatives and the shapes of curves.

The mean value theorem. If f is a differentiable function on the interval $[a, b]$, then there exist a number c , $a < c < b$, such that $f'(c) = \frac{f(b) - f(a)}{b - a}$ or $f(b) - f(a) = f'(c)(b - a)$.

Increasing / decreasing test.

- (a.) If $f'(x) > 0$ on an interval, then f is increasing on that interval
- (b.) If $f'(x) < 0$ on an interval, then f is decreasing on that interval

The first derivative test. Suppose that c is a critical number of a continuous function f .

- (a.) If f' changes from positive to negative at c , then f has a local max at c .
- (b.) If f' changes from negative to positive at c , then f has a local min at c .
- (c.) If f' does not change sign at c , then f has no local max or min at c .

A function is called *concave upward* (CU) on an interval I if f' is an increasing function on I . It is called *concave downward* (CD) on I if f' is an decreasing on I .

A point where a curve changes its direction of concavity is called an *inflection point*.

Concavity test.

- (a.) If $f''(x) > 0$ on an interval, then f is CU on this interval.
- (b.) If $f''(x) < 0$ on an interval, then f is CD on this interval.

The second derivative test. Suppose f'' is continuous near c .

- (a.) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local min at c .
- (b.) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local max at c .

Example 11. For the function $y = x^2e^x$ find

- (a.) All asymptotes.

(b.) Intervals on which the function is increasing/decreasing.

(c.) All local minima/local maxima, absolute minima/absolute maxima.

(d.) Intervals on which the function is CU/CD.

(e.) Inflection points.

Section 5.5 Applied maximum and minimum problems

Steps in solving applied max and min problems

1. Understand the problem.
2. Draw a diagram.
3. Introduce notation. Assign a symbol to the quantity that is to be minimized or maximized (let us call it Q). Also select symbols (a, b, c, \dots, x, y) for other unknown quantities and label the diagram with these symbols.
4. Express Q in terms of some of the other symbols from step 3.
5. If Q has been expressed as a function of more than one variable in step 4, use the given information to find relationships (in the form of equation) among these variables. Then use these equations to eliminate all but one of the variables in the expression for Q . Thus, Q will be given as a function of one variable.
6. Find the *absolute* maximum or minimum of Q .

Example 12. A cylindrical can without a top is made to contain V cm³ of liquid. Find the dimensions that will minimize the cost of the metal to make the can.

Section 5.7 Antiderivatives

Definition. Function $F(x)$ is called an **antiderivative** of $f(x)$ on an interval I if $F'(x) = f(x)$ for all $x \in I$.

Theorem. If F is an antiderivative of f on an interval I , then the most general antiderivative of f on I is $\boxed{F(x) + C}$ where C is a constant.

Table of antidifferentiation formulas

<i>Function</i>	<i>Antiderivative</i>
$af(x)$, a is a constant	$aF(x) + C$
$f(x) + g(x)$	$F(x) + G(x) + C$
a , a is a constant	$ax + C$
x	$\frac{x^2}{2} + C$
$x^n, n \neq -1$	$\frac{x^{n+1}}{n+1} + C$
$\frac{1}{x}$	$\ln x + C$
e^x	$e^x + C$
a^x	$\frac{a^x}{\ln a} + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$\sec^2 x$	$\tan x + C$
$\csc^2 x$	$-\cot x + C$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + C$
$\frac{1}{1+x^2}$	$\tan^{-1} x + C$

Example 12. Find the most general antiderivative of the function

(a.) $f(x) = (\sqrt{x} + 1)(x - \sqrt{x} + 1)$

(b.) $f(x) = \sin x + \frac{2}{1+x^2} + \frac{3}{\sqrt{1-x^2}}$

Rectilinear motion.

If the object has a position function $s = s(t)$, then $v(t) = s'(t)$ (the position function is an antiderivative for the velocity function),

$a(t) = v'(t)$ (the velocity function is an antiderivative to the acceleration function)

Example 13. A particle is moving with the acceleration $a(t) = t^2 - t$, $s(0) = 0$, $v(0) = 1$. Find the position of the particle.

Antiderivatives of vector functions.

Definition. A vector function $\vec{R}(t) = \langle X(t), Y(t) \rangle$ is called **an antiderivative** of $\vec{r}(t) = \langle x(t), y(t) \rangle$ on an interval I if $\vec{R}'(t) = \vec{r}(t)$ that is, $X'(t) = x(t)$ and $Y'(t) = y(t)$.

Theorem. If \vec{R} is an antiderivative of \vec{r} on an interval I , then the most general antiderivative of \vec{r} on I is $\boxed{\vec{R} + \vec{C}}$ where \vec{C} is an arbitrary constant vector.

Example 14. Find the vector-function that describe the position of particle that has an acceleration $\vec{a}(t) = 2t\vec{i} + 3\vec{j}$, $\vec{v}(0) = \vec{i} - \vec{j}$, and initial position at $(1,2)$.

Section 6.1 Sigma notation

Definition. If $a_m, a_{m+1}, a_{m+2}, \dots, a_n$ are real numbers and m and n are integers such that $m \leq n$, then

$$a_m + a_{m+1} + a_{m+2} + \dots + a_n = \sum_{i=m}^n a_i$$

Theorem. If c is any constant (this means that c does not depend on i), then

$$(a.) \sum_{i=1}^n ca_i = c \sum_{i=1}^n a_i \quad (b.) \sum_{i=1}^n (a_i \pm b_i) = \sum_{i=1}^n a_i \pm \sum_{i=1}^n b_i$$

$$\sum_{i=1}^n 1 = n$$

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^n i^3 = \left[\frac{n(n+1)}{2} \right]^2$$

Example 15. Find the value of the sum $\sum_{i=0}^5 i(i-1)$.