Chapter 2. **Limits and rates of change**
Section 2.2. **The limit of the function**

**Definition.** We write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as $x$ approaches $a$, equals $L$" if we can make values of $f(x)$ arbitrary close to $L$ by taking $x$ to be sufficiently close to $a$ but not equal to $a$.

**Definition.** We write

$$\lim_{x \to a^-} f(x) = L$$

and say the **left-handed limit of** $f(x)$ **as** $x$ **approaches** $a$ **(or the limit of** $f(x)$ **as** $x$ **approaches** $a$ **from the left)**, equals $L$ if we can make values of $f(x)$ arbitrary close to $L$ by taking $x$ to be sufficiently close to $a$ and $x < a$.

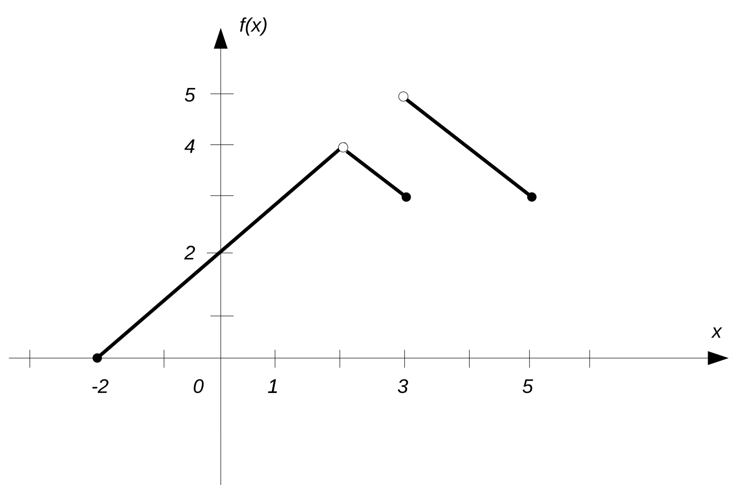
**Definition.** We write

$$\lim_{x \to a^+} f(x) = L$$

and say the **right-handed limit of** $f(x)$ **as** $x$ **approaches** $a$ **(or the limit of** $f(x)$ **as** $x$ **approaches** $a$ **from the right)**, equals $L$ if we can make values of $f(x)$ arbitrary close to $L$ by taking $x$ to be sufficiently close to $a$ and $x > a$.

$\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$

**Example 1.** Given the graph of the function $f$




Find:
(a.) $\lim_{x \to 1} f(x)$

(b.) $\lim_{x \to 2^+} f(x)$

(c.) $\lim_{x \to 2^-} f(x)$

(d.) $\lim_{x \to 3^+} f(x)$

(e.) $\lim_{x \to 3^-} f(x)$

**Definition.** Let $f$ be a function defined on both sides of $a$, except, possibly at $a$ itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrary large by taking $x$ to be sufficiently close to $a$ but not equal to $a$.

**Definition.** Let $f$ be a function defined on both sides of $a$, except, possibly at $a$ itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrary large negative by taking $x$ to be sufficiently close to $a$ but not equal to $a$.

**Definition.** The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\begin{array}{ccc} \lim_{x \to a} f(x) = \infty & \lim_{x \to a^+} f(x) = \infty & \lim_{x \to a^-} f(x) = \infty \\ \lim_{x \to a} f(x) = -\infty & \lim_{x \to a^+} f(x) = -\infty & \lim_{x \to a^-} f(x) = -\infty \end{array}$$

**Example 2.** Find

(a.) $\lim_{x \to 4^+} \dfrac{5}{x-4}$

(b.) $\lim_{x \to 4^-} \dfrac{5}{x-4}$

(c.) $\lim_{x \to 4} \dfrac{5}{x-4}$

**Definition.** We write

$$\lim_{t \to a} \vec{r}(t) = \vec{b}$$

and say "the limit of $\vec{r}(t)$, as $t$ approaches $a$, equals $\vec{b}$" if we can make vector $\vec{r}(t)$ arbitrary close to $\vec{b}$ by taking $t$ to be sufficiently close to $a$ but not equal to $a$.

If $\vec{r}(t) =< f(t), g(t) >$, then

$$\lim_{t \to a} \vec{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t) \right\rangle$$

provided the limits of the component functions exist.