Chapter 4. **Inverse functions: exponential, logarithmic, and inverse trigonometric functions**

Section 4.1 **Exponential functions and their derivatives**

An **exponential function** is a function of the form

$$f(x) = a^x$$

where a is a positive constant. It is defined in five stages:

- If $x = n$, a positive integer, then

$$a^n = \underbrace{a \cdot a \cdot \dots \cdot a}_{n \text{ times}}$$

- $a^0 = 1$
- If $x = -n$, n is a positive integer, then

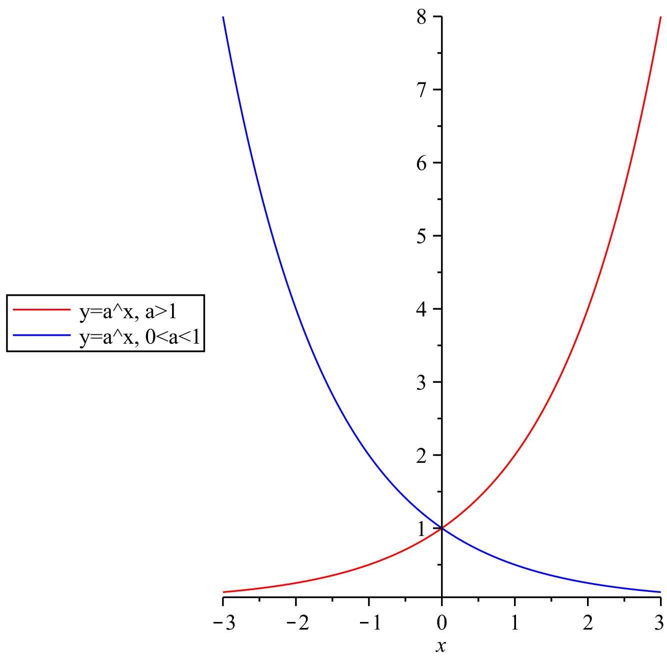
$$a^{-n} = \frac{1}{a^n}$$

- If $x = \frac{p}{q}$ is a rational number, where p and q are integers and $q > 0$, then

$$a^{p/q} = \sqrt[q]{a^p}$$

- If x is an irrational number, then

$$a^x = \lim_{r \to x} a^r \qquad r \text{ rational}$$



Theorem. If $a > 0$ and $a \neq 1$, then $f(x) = a^x$ is continuous function with domain $(-\infty, \infty)$ and range $(0, \infty)$.

- If $0 < a < 1$, $f(x) = a^x$ is decreasing function
- if $a > 1$, $f(x) = a^x$ is increasing function
- If $a, b > 0$ and x, y are reals, then

$$1.\ a^{x+y} = a^x a^y \quad 2.\ a^{x-y} = \frac{a^x}{a^y} \quad 3.\ (a^x)^y = a^{xy} \quad 4.\ (ab)^x = a^x b^x$$

- If $0 < a < 1$, $\lim_{x\to-\infty} a^x = \infty$, $\lim_{x\to\infty} a^x = 0$
- If $a > 1$, $\lim_{x\to-\infty} a^x = 0$, $\lim_{x\to\infty} a^x = \infty$

Example 1. Find each limit.

1. $\lim_{x\to\infty} (1.1)^x$

2. $\lim_{x\to-\infty} \pi^x$

3. $\lim_{x\to-\infty} \frac{2^{3x} - 2^{-3x}}{2^{3x} + 2^{-3x}}$

Derivative of exponential function. If $f(x) = a^x$, then

$$f'(x) = (a^x)' = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h\to 0} \frac{a^x a^h - a^x}{h} = \lim_{h\to 0} \frac{a^x(a^h - 1)}{h} = a^x \lim_{h\to 0} \frac{a^h - 1}{h} = a^x f'(0)$$

Let e be a number such that

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

$$e \approx 2.7182818284590452$$

Then

$$\frac{d}{dx}e^x = e^x$$

Example 2. Differentiate each function.

1. $f(x) = e^{\sqrt{x}}$

2. $f(x) = xe^{-x^2}$

3. $f(x) = e^{x \tan n}$

4. $f(x) = x^e$

Example 3. Show that the function $y = e^{2x} + e^{-3x}$ satisfy the differential equation

$$y'' + y' - 6y = 0$$