Section 4.8 **Indeterminate forms and L'Hospitale's rule**

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ for points close to a (except, possibly a). Suppose that
$\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \left| \frac{0}{0} \text{ or } \frac{\infty}{\infty} \right| = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Example 1. Evaluate $\lim_{x\to 0} \frac{x\cos x - \sin x}{x^3}$

Indeterminate products If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$, then

$$\lim_{x\to a} f(x)g(x) = |\infty \cdot 0| = \lim_{x\to a} \frac{f(x)}{1/g(x)} = \lim_{x\to a} \frac{g(x)}{1/f(x)} = \left| \frac{0}{0} \text{ or } \frac{\infty}{\infty} \right|$$

and now we can use L'Hospital's Rule.

Example 2. Evaluate $\lim_{x\to 1} (1-x)\tan\frac{\pi x}{2}$

Indeterminate differences If we have to find $\lim\limits_{x\to a}(f(x)-g(x)) = |\infty-\infty|$ ($\lim\limits_{x\to a} f(x) = \infty, \lim\limits_{x\to a} g(x) = \infty$), then we have to convert the difference into a quotient (by using a common denominator or rationalization, or factoring out a common factor) so that we have an indeterminate form of type $\left|\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right|$ and we can use L'Hospital's Rule.

Example 3. Evaluate $\lim\limits_{x\to 1}\left(\dfrac{x}{x-1}-\dfrac{1}{\ln x}\right)$

Indeterminate powers $\lim\limits_{x\to a}[f(x)]^{g(x)} = |0^0 \text{ or } \infty^0 \text{ or } 1^\infty| = \lim\limits_{x\to a} \mathrm{e}^{g(x)\ln f(x)} = \mathrm{e}^{\lim\limits_{x\to a}[g(x)\ln f(x)]}$.
Now let's find

$$\lim_{x\to a}[g(x)\ln f(x)] = |0\cdot\infty| = \lim_{x\to a}\frac{\ln f(x)}{\frac{1}{g(x)}} = \left|\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right| = b$$

Then
$\lim\limits_{x\to a}[f(x)]^{g(x)} = \mathrm{e}^b$

Example 4. Evaluate $\lim\limits_{x\to 0}(1+x^2)^{\frac{1}{x}}$.