Tuesday, Nov. 29, 7:30-9:30 PM in ARCC 105.
long green scantron 822-E.
over 4.2-4.6, 4.8, 5.1-5.3, 5.5, 5.7

1. A sample of radium initially has a mass of 100 mg. After 1500 years, the mass has been reduced to 30 mg. What is the size of the mass after 350 years? Assume the size of the mass follows the law of exponential decay.

   (a) $100\left(\frac{10}{3}\right)^{7/30}$ mg
   (b) $100\left(\frac{10}{3}\right)^{30/7}$ mg
   (c) $100\left(\frac{3}{10}\right)^{30/7}$ mg
   (d) $100\left(\frac{3}{10}\right)^{7/30}$ mg
   (e) None of these

$m(t) = m(0)e^{kt}$

$m(0)=100$, $m(1500)=30$

$m(350)=?$

$m(t) = 100e^{kt}$

$m(1500) = 10\cancel{0}e^{1500k} = 3\cancel{0}$

$e^{1500k} = \frac{3}{10}$

$1500k = \ln\frac{3}{10}$

$k = \frac{1}{1500}\ln\left(\frac{3}{10}\right)$

$m(t) = 100e^{\frac{t}{1500}\ln\left(\frac{3}{10}\right)}$

$= 100\left(\frac{3}{10}\right)^{\frac{t}{1500}}$

$m(350) = 100\left(\frac{3}{10}\right)^{\frac{350}{1500}}$

$= \boxed{100\left(\frac{3}{10}\right)^{\frac{7}{30}}}$

2. If $f'(x) = 2\sin x + 4\cos x - e^x$ and $f(0) = 5$, what is $f(\pi)$?

   (a) $10 - e^{\pi}$
   (b) $4 - e^{\pi}$
   (c) $7 - e^{\pi}$
   (d) $3 - e^{\pi}$
   (e) $6 - e^{\pi}$

$f(x) = -2\cos x + 4\sin x - e^x + C$

$f(0) = -2\cos 0 + 4\sin 0 - e^0 + C = 5$

$C - 3 = 5$

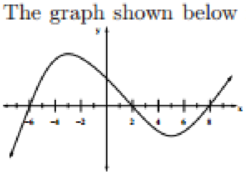
$C = 8$

$f(x) = -2\cos x + 4\sin x - e^x + 8$

$f(\pi) = -2\cos\pi + 4\sin\pi - e^{\pi} + 8$

$= 2 + 8 - e^{\pi} = 10 - e^{\pi}$

3. The graph shown below is the graph of the second derivative of $f(x)$, that is $f''(x)$. Where is $f(x)$ concave up?



f is CU when $f''(x) > 0$

$(-6, 2) \cup (8, \infty)$

   (a) $(1, \infty)$
   (b) $(-\infty, -3) \cup (3, \infty)$
   (c) $(-\infty, 1)$
   (d) $(-\infty, -6) \cup (2, 8)$
   (e) $(-6, 2) \cup (8, \infty)$

4. Find the value of the limit: $\lim_{x\to 0} \frac{2^{3x}-5^x}{4x}$.

(a) 0

(b) $\frac{1}{4}\ln\left(\frac{8}{5}\right)$

(c) $\frac{1}{4}\ln\left(\frac{2}{5}\right)$

(d) 1

(e) $\frac{1}{4}\ln 10$

$\left|\frac{0}{0}\right| = \lim_{x\to 0} \frac{2^{3x}\ln 2\,(3) - 5^x \ln 5\,(1)}{4}$

$= \lim_{x\to 0} \frac{3(2^{3x})\ln 2 - 5^x \ln 5}{4} = \frac{3\ln 2 - \ln 5}{4}$

$= \frac{\ln 2^3 - \ln 5}{4} = \frac{1}{4}\ln\frac{8}{5}$

5. Find all critical numbers for $f(x) = \sqrt[3]{x^2 - x - 2}$ $= (x^2-x-2)^{1/3}$

(a) $x = 2$ and $x = -1$

(b) $x = \frac{1}{2}$

(c) $x = 2$, $x = -1$ and $x = \frac{1}{2}$

(d) $x = -2$, $x = 1$ and $x = 2$

(e) $x = -2$, $x = 1$ and $x = \frac{1}{2}$

$f'(x) = \frac{1}{3}(x^2-x-2)^{-2/3}(2x-1) = \frac{2x-1}{3\sqrt[3]{(x^2-x-2)}}$

Critical numbers: $2x-1=0 \Rightarrow x = 1/2$

$x^2-x-2=0 \Rightarrow x=-1,\ x=2$

$(x-2)(x+1)=0$

6. If $g(x)$ is the inverse of $f(x) = \sqrt{x^3+3x+2}$, what is $g'(4)$?

(a) $\frac{15}{8}$

(b) $\frac{1}{8}$

(c) $\frac{15}{4}$

(d) $\frac{9}{4}$

(e) $\frac{8}{15}$

$g'(4) = \frac{1}{f'(g(4))}$, $g(4)=x \iff f(x)=4$

$\sqrt{x^3+3x+2} = 4$

$x^3+3x+2=16$

$\boxed{x=2:}$ $8+6+2=16$

$g(4)=2.$

$f'(x) = \frac{1}{2}(x^3+3x+2)^{-1/2}(3x^2+3)$

$f'(2) = \frac{1}{2}(8+6+2)^{-1/2}(3(4)+3) = \frac{15}{8}$

$g'(4) = \frac{1}{f'(2)} = \frac{8}{15}$

7. Which of the following expressions is equivalent to $\cos(\underbrace{\arctan x}_{y})$?

(a) $\dfrac{x}{\sqrt{1-x^2}}$

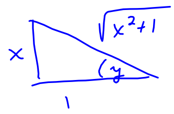
(b) $\dfrac{\sqrt{1-x^2}}{x}$

(c) $\sqrt{x^2+1}$

(d) $\dfrac{1}{\sqrt{x^2+1}}$

(e) $\dfrac{x}{\sqrt{x^2+1}}$

$\arctan x = y \Leftrightarrow \tan y = x$



$\cos y = \dfrac{1}{\sqrt{x^2+1}}$

8. An object is traveling at a speed of 60 m/s when the brakes are fully applied, producing a constant deceleration of 12 meters per second squared. What is the distance covered before the object comes to a stop?

(a) 450 meters

(b) 210 meters

(c) 200 meters

(d) 150 meters

(e) 310 meters

$a = -12$

$v(t) = -12t + C_1, \quad v(0) = 60$

$v(0) = -12(0) + C_1 = 60$

$v(t) = -12t + 60$

$s(t) = -12\dfrac{t^2}{2} + 60t + C_2$

$= -6t^2 + 60t + C_2, \quad s(0) = 0, \quad C_2 = 0$

$s(t) = -6t^2 + 60t$

stop when $v(t) = 0$ or $-12t + 60 = 0$

$t = 5$

$s(5) = -6(25) + 60(5) = 300 - 150 = \boxed{150}$

9. What is the domain of $f(x) = \ln(1 - \ln x)$?

(a) $(0, \infty)$

(b) $(1, \infty)$

(c) $(0, e)$

(d) $(1, e)$

(e) (e, ∞)

$\ln(1-\ln x)$: $1 - \ln x > 0 \Rightarrow \ln x < 1 \Rightarrow x < e$

$\ln x$: $x > 0$

$\boxed{(0, e)}$

Domain: $5-x>0$ $\quad x<5$
$2-x>0$ $\quad x<2$
$\boxed{(-\infty, 2)}$

2 $\quad$ 5

10. Solve for x: $\log_{10}(5-x)+\log_{10}(2-x)=1$.

~~(a)~~ $x=3$
(b) $x=0$
~~(c)~~ $x=7$
~~(d)~~ $x=0,\ x=7$
(e) $x=0,\ x=-\dfrac{2}{3}$

$$\log_{10}(5-x)(2-x)=1$$
$$(5-x)(2-x)=10$$
$$10-5x-2x+x^2=10$$
$$x^2-7x=0$$
$$x(x-7)=0$$
$$x_1=0,\quad x_2=7.$$

11. Find the y intercept of the line tangent to the graph of $f(x)=(\ln x)^2$ at $x=e$.

(a) $y=e+1$
(b) $y=-1$
(c) $y=-2$
(d) $y=1-2e$
(e) $y=1$

$$f'(x)=2\ln x\,\frac{1}{x}=\frac{2\ln x}{x}$$
$$f'(e)=\frac{2\ln e}{e}=\frac{2}{e}$$
$$f(e)=(\ln e)^2=1$$
$$y=\frac{2}{e}(x-e)+1$$
$$y=\frac{2}{e}x-1;\quad x=0,\ \text{then}\quad y(0)=-1$$

12. Find the intervals where $f(x)=\dfrac{\ln x}{x}$ is increasing or decreasing.

(a) $f(x)$ is increasing on the interval $\left(0,\dfrac{1}{e}\right)$ and decreasing on the interval $\left(\dfrac{1}{e},\infty\right)$.
(b) $f(x)$ is increasing on the interval (e,∞) and decreasing on the interval $(0,e)$.
(c) $f(x)$ is increasing on the interval $\left(\dfrac{1}{e},\infty\right)$ and decreasing on the interval $\left(0,\dfrac{1}{e}\right)$.
(d) $f(x)$ is increasing on the interval $(0,e)$ and decreasing on the interval (e,∞).
(e) $f(x)$ is always increasing.

$$f'(x)=\frac{\frac{1}{x}x-\ln x}{x^2}=\frac{1-\ln x}{x^2}>0$$
$$1-\ln x=0$$
$$\ln x=1$$
$$x=e$$
$$x>0\ (\text{domain})$$

$+$ $\quad$ $-$
0 $\quad$ e

13. Find the absolute maximum and absolute minimum for $f(x) = x^2 + \frac{2}{x}$ over the interval $\left[\frac{1}{2}, 2\right]$.

(a) Absolute maximum: $y = 5$, Absolute minumum: $y = 3$

(b) Absolute maximum: $y = 4$, Absolute minumum: $y = 3$

(c) Absolute maximum: $y = 5$, Absolute minumum: $y = -1$

(d) Absolute maximum: $y = 4.5$, Absolute minumum: $y = -1$

(e) Absolute maximum: $y = 4.5$, Absolute minumum: $y = 3$

$f'(x) = 2x - \frac{2}{x^2} = 0$

$\frac{2x^3-2}{x^2} = 0$, $x \neq 0$.

$2x^3 - 2 = 0$ or $x^3 - 1 = 0$

or $x = 1$.

$f(1/2) = \frac{1}{4} + \frac{2}{1/2} = 4 + \frac{1}{4} = \frac{17}{4}$

$f(1) = 3 = \frac{12}{4}$ abs min

$f(2) = 4 + \frac{2}{2} = 5 = \frac{20}{4}$ abs max

14. If $f(x) = \arctan(x^2 + 2x + 3) + \arccos(3x)$, then $f'(0) =$

(a) $\frac{2}{\sqrt{10}} - 3$

(b) $\frac{16}{5}$

(c) $\frac{1}{\sqrt{10}} - 3$

(d) $-\frac{14}{5}$

(e) $-\frac{2}{5}$

$f'(x) = \frac{1}{1+(x^2+2x+3)^2}(x^2+2x+3)' - \frac{1}{\sqrt{1-(3x)^2}}(3)$

$f'(x) = \frac{2x+2}{1+(x^2+2x+3)^2} - \frac{3}{\sqrt{1-9x^2}}$

$f'(0) = \frac{2}{1+3^2} - \frac{3}{1} = \frac{2}{10} - 3 = \frac{1}{5} - 3 = -\frac{14}{5}$

15. Find $\frac{dy}{dx}$ if $y = (\tan x)^{\sqrt{3x}}$

(a) $\frac{dy}{dx} = (\tan x)^{\sqrt{3x}}\left(\frac{3\ln(\tan x)}{2\sqrt{3x}} + \frac{\sqrt{3x}\sec^2 x}{\tan x}\right)$

(b) $\frac{dy}{dx} = (\tan x)^{\sqrt{3x}}\left(\frac{3\ln(\tan x)}{2\sqrt{3x}} + \sqrt{3x}(\sec x)\right)$

(c) $\frac{dy}{dx} = (\tan x)^{\sqrt{3x}}\left(\frac{3\ln(\tan x)}{2\sqrt{3x}} + \sqrt{3x}(\cot x)\right)$

(d) $\frac{dy}{dx} = (\tan x)^{\sqrt{3x}}\left(\frac{\ln(\tan x)}{2\sqrt{3x}} + \frac{\sqrt{3x}\sec^2 x}{\tan x}\right)$

(e) $\frac{dy}{dx} = (\tan x)^{\sqrt{3x}}\left(\frac{\ln(\tan x)}{2\sqrt{3x}} + \sqrt{3x}(\cot x)\right)$

Logarithmic differentiation.

$\ln y = \sqrt{3x}\ \ln(\tan x)$

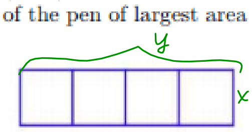
$\frac{y'}{y} = \sqrt{3} \cdot \frac{1}{2} x^{-1/2} \ln(\tan x) + \sqrt{3x}\ \frac{1}{\tan x}(\sec^2 x)$

$y' = y\left(\frac{\sqrt{3}\ln(\tan x)}{2\sqrt{x}} + \frac{\sqrt{3x}\sec^2 x}{\tan x}\right)$

PART II: Work Out

Directions: Present your solutions in the space provided. *Show all your work* neatly and concisely and *Box your final answer*. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

16. (8 pts) You have 100 feet of fencing to construct a pen with four equal sized stalls as shown. What are the dimensions of the pen of largest area and what is the largest area?



Perimeter: $P = 5x + 2y = 100$

$$y = \frac{100-5x}{2} = 50 - \frac{5}{2}x$$

area: $A = xy = x\left(50 - \frac{5}{2}x\right) = 50x - \frac{5}{2}x^2$

$$A' = 50 - \frac{5}{2}(2x) = 0$$

$$50 - 5x = 0 \Rightarrow x = 10 \text{ ft}$$

$$y = 50 - 25 = 25 \text{ ft}$$

$\boxed{10 \times 25}$ $\boxed{A = 250 (ft^2)}$ – max.

$$A'' = -5 < 0.$$

17. (6 pts) Find the inverse of $f(x) = \frac{3e^x}{2+e^x}$.

$$y = \frac{3e^x}{2+e^x}$$

$$(2+e^x)y = 3e^x$$

$$2y + ye^x = 3e^x$$

$$3e^x - ye^x = 2y$$

$$e^x(3-y) = 2y$$

$$e^x = \frac{2y}{3-y}, \quad x = \ln \frac{2y}{3-y} = f^{-1}(y)$$

$$f^{-1}(x) = \boxed{\ln \frac{2x}{3-x}}$$

18. Given $f(x) = 2xe^{3x}$:

(i) (3 pts) Find the intervals where is $f(x)$ increasing or decreasing.

$f'(x) = 2e^{3x} + 2x\,(3e^{3x}) = e^{3x}(2+6x) = 0$

$x = -\frac{2}{6} = -\frac{1}{3}$

$f'(0) = 2 > 0$

$f'(-1) = e^{-3}(2-6) < 0$

$-$ $\quad$ $+$

$-1/3$

f is decreasing on $(-\infty, -1/3)$

f is increasing on $(-1/3, \infty)$

(ii) (2 pts) Find the local maximim and/or minimum for $f(x)$, if any.

local min @ $x = -1/3$.

$f(-1/3) = -\frac{2}{3}e^{-1}$

$\boxed{\text{local min @ } \left(-\frac{1}{3}, -\frac{2}{3e}\right)}$

no local max

(iii) (3 pts) Find the intervals of concavity for $f(x)$. $\quad f'(x) = (2+6x)e^{3x}$

$f''(x) = 6e^{3x} + (2+6x)(3)e^{3x}$

$= e^{3x}(6+6+18x) = e^{3x}(12+18x) = 0$

$x = -\frac{12}{18} = -\frac{2}{3}$

$\cap$ $\quad$ $\cup$

$-$ $\quad$ $+$

$-2/3$

f is CD on $(-\infty, -2/3)$

f is CU on $(-2/3, \infty)$

(iv) (2 pts) Find the point(s) of inflection for $f(x)$, if any. @ $x = -2/3$

$f(-\frac{2}{3}) = 2(-\frac{2}{3})e^{3(-\frac{2}{3})} = -\frac{4}{3}e^{-2}$

inflection point @ $\boxed{\left(-\frac{2}{3}, -\frac{4}{3e^2}\right)}$

[redacted]

19. (8 pts) Find $\lim_{x\to\infty}\left(1-\frac{2}{x}\right)^x$ $= |1^\infty| = \lim_{x\to\infty} e^{x\ln(1-\frac{2}{x})} = e^{\lim_{x\to\infty} x\ln(1-\frac{2}{x})} = \boxed{e^{-2}}$

$$\lim_{x\to\infty} x\ln\left(1-\frac{2}{x}\right) \quad |\infty\cdot 0| = \lim_{x\to\infty}\frac{\ln\left(1-\frac{2}{x}\right)}{\frac{1}{x}} \quad \left|\frac{0}{0}\right|$$

$$= \lim_{x\to\infty}\frac{\frac{1}{1-\frac{2}{x}}\left(1-\frac{2}{x}\right)'}{-1/x^2} = \lim_{x\to\infty}\frac{\frac{1}{1-2/x}\left(\frac{2}{\cancel{x^2}}\right)}{-1/\cancel{x^2}}$$

$$= -2\lim_{x\to\infty}\frac{1}{1-\underbrace{2/x}_{\to 0}} = -2$$

$$\boxed{e^{-2}}$$

20. (8 pts) Sketch a curve satisfying the following conditions.

(i) The domain of $f(x)$ is all real numbers.

(ii) $f(2) = -2$, $f(0) = 0$, $f'(2) = 0$ ~~local min~~ or local max.

(iii) $f'(x) < 0$ if $0 < x < 2$, $f'(x) > 0$ if $x > 2$

(iv) $f''(x) < 0$ if $0 \leq x < 1$ or if $x > 4$

(v) $f''(x) > 0$ if $1 < x < 4$

(vi) $\lim_{x \to \infty} f(x) = 2$

(vii) The graph of $f(x)$ is symmetric about the y axis.

f is decreasing on $(0,2)$
f is increasing on $(2,\infty)$
f is CD on $(0,1)$ and $(4,\infty)$
f is CU on $(1,4)$
$y=2$ is H.A.