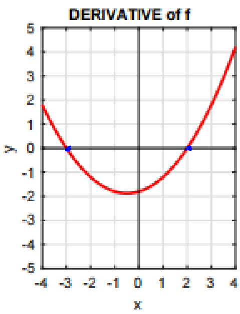
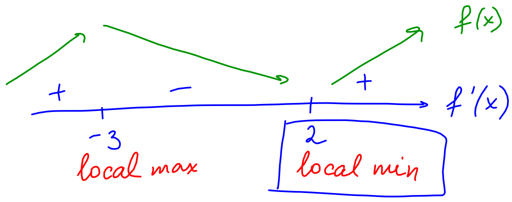
1. Below is the graph of $f'(x)$, the derivative of $f(x)$. At what value(s) of x does $f(x)$ attain a local minimum?





(a) $x = -\frac{1}{2}$

(b) $x = 2$

(c) $x = -3$ and $x = 2$

(d) $x = -\frac{1}{2}$ and $x = 2$

(e) $x = -3$

2. $\lim_{x \to 3^+} \ln\left(\frac{x-3}{x}\right) =$ $\lim_{y \to 0^+} \ln y = \boxed{-\infty}$

$\lim_{x \to 3^+} \frac{x-3}{x} = \frac{3-3}{3} = 0$

(a) 1

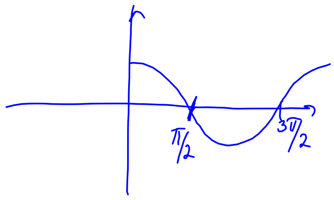
(b) 0

(c) ∞

(d) e

(e) $-\infty$

3. Find all critical numbers for $f(x) = \sin^2 x + \cos x$, $0 < x < 2\pi$.



$f'(x) = 2\sin x \cos x - \sin x = 0$

$\sin x(2\cos x - 1) = 0$

$\sin x = 0$ or $\cos x = \frac{1}{2}$

$x = \pi$

$x = \frac{\pi}{3}, \ 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

(a) $x = \pi$, $x = \frac{\pi}{3}$, $x = \frac{5\pi}{3}$

(b) $x = \frac{\pi}{3}$, $x = \frac{5\pi}{3}$

(c) $x = \pi$, $x = \frac{2\pi}{3}$, $x = \frac{4\pi}{3}$

(d) $x = \frac{2\pi}{3}$, $x = \frac{4\pi}{3}$

(e) $x = \pi$, $x = \frac{\pi}{6}$, $x = \frac{11\pi}{6}$

4. The number of bacteria in a culture is increasing according to the Law of Exponential Growth. If there are initially 125 bacteria in the culture and there are 350 bacteria after 2 hours, how many bacteria are present after 24 hours?

(a) $125\left(\frac{5}{14}\right)^{12}$

(b) $125\left(\frac{14}{5}\right)^{24}$

(c) $\frac{125}{2}\left(\frac{14}{5}\right)^{24}$

(d) $125\left(\frac{14}{5}\right)^{12}$

(e) $\frac{125}{2}\left(\frac{5}{14}\right)^{12}$

$p(t)$ population after t hours

$p(t) = p(0)e^{kt}$

$p(0) = 125, \quad p(2) = 350, \quad$ Find $p(24)$.

$p(t) = 125e^{kt}$

$p(2) = 125e^{2k} = 350$

$e^{2k} = \frac{350}{125} = \frac{7 \cdot 50}{5 \cdot 25} = \frac{70}{25} = \frac{14}{5}$

$2k = \ln\frac{14}{5}, \quad k = \frac{1}{2}\ln\frac{14}{5}$

$p(t) = (125)\, e^{\frac{t}{2}\ln\frac{14}{5}} = (125)\left(\frac{14}{5}\right)^{t/2}$

$p(24) = \boxed{(125)\left(\frac{14}{5}\right)^{12}}$

5. What is the slope of the curve $f(x) = 3x\arcsin(x)$ at $x = -\frac{1}{2}$?

(a) $m = -\pi - \frac{3}{\sqrt{3}}$

(b) $m = -\frac{\pi}{2} - \frac{3}{\sqrt{3}}$

(c) $m = -\frac{\pi}{2} - \frac{3}{\sqrt{5}}$

(d) $m = \frac{7\pi}{2} - \frac{3}{\sqrt{3}}$

(e) $m = -\pi - \frac{3}{\sqrt{5}}$

$f'(x) = 3\arcsin x + \frac{3x}{\sqrt{1-x^2}}$

$f'\left(-\frac{1}{2}\right) = 3\arcsin\left(-\frac{1}{2}\right) + \frac{3(-1/2)}{\sqrt{1-\left(-\frac{1}{2}\right)^2}}$

$= 3\left(-\frac{\pi}{6}\right) + \frac{-3/2}{\sqrt{3/4}} = -\frac{\pi}{2} - \frac{3}{\sqrt{3}}$

6. Which of the following is equivalent to $2\ln(x^2+3) - \ln(x) - \frac{1}{2}\ln(x^2+1)$?

(a) $\ln\left(\frac{(x^2+3)^2}{x\sqrt{x^2+1}}\right)$

(b) $\ln\left(\frac{(x^2+3)^2\sqrt{x^2+1}}{x}\right)$

(c) $\ln\left(\frac{x(x^2+3)^2}{\sqrt{x^2+1}}\right)$

(d) $\ln\left(\frac{4(x^2+3)}{x(x^2+1)}\right)$

(e) $\frac{\ln(x^2+3)^2}{\ln x\sqrt{x^2+1}}$

$= \ln(x^2+3)^2 - \ln x - \ln(x^2+1)^{1/2}$

$= \boxed{\ln\frac{(x^2+3)^2}{x\sqrt{x^2+1}}}$

7. $\lim_{x\to 0} \frac{\arctan(2x)}{3x} =$ $\frac{0}{0}$ $= \lim_{x\to 0} \frac{\frac{1}{1+(2x)^2}(2)}{3} = \boxed{\frac{2}{3}}$

(a) 0

(b) 1

(c) $\frac{2}{3}$

(d) 6

(e) $\frac{1}{3}$

8. Find the absolute extrema for $f(x) = 3x - x^3$ on $[0, 3]$.

(a) Absolute maximum is 2, Absolute minimum is -18

(b) Absolute maximum is 0, Absolute minimum is -18

(c) Absolute maximum is 2, Absolute minimum is -2

(d) Absolute maximum is 2, Absolute minimum is 0

(e) Absolute maximum is 4, Absolute minimum is -2

$f'(x) = 3 - 3x^2 = 0$

$x^2 = 1, \quad x = 1$ or $\boxed{x = -1}$ not in $[0,3]$

$f(0) = 0$

$f(1) = 3 - 1 = \boxed{2 \text{ abs max}}$

$f(3) = 3(3) - 3^3 = 9 - 27 = \boxed{-18 \text{ abs min}}$

9. Find $f'(x)$ for $f(x) = x^{\ln x}$.

(a) $f'(x) = x^{\ln x}\left(\frac{2\ln x}{x}\right)$

(b) $f'(x) = x^{\ln x}\left(\frac{\ln x}{x}\right)^2$

(c) $f'(x) = x^{\ln x}\left(\frac{1}{x^2}\right)$

(d) $f'(x) = \frac{2\ln x}{x}$

(e) $f'(x) = \ln x(x^{\ln x - 1})$

$\ln f = \ln\left(x^{\ln x}\right)$

$= \ln x(\ln x)$

$\ln f = (\ln x)^2$

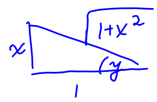
$\frac{f'}{f} = 2\ln x \, \frac{1}{x}$

$f' = f \cdot \frac{2\ln x}{x}$

$= x^{\ln x} \cdot \frac{2\ln x}{x}$

10. Which of the following is equivalent to $\sec(\arctan x)$? $= \dfrac{1}{\cos(\arctan x)} = \dfrac{1}{\cos y} = \boxed{\sqrt{1+x^2}}$

$y = \arctan x \Leftrightarrow x = \tan y$



$\cos y = \dfrac{1}{\sqrt{1+x^2}}$

(a) $\dfrac{1}{\sqrt{x^2-1}}$

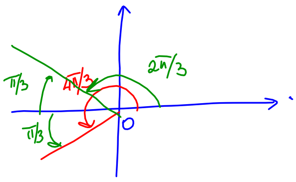
(b) $\sqrt{x^2+1}$

(c) $\dfrac{1}{\sqrt{x^2+1}}$

(d) $\sqrt{x^2-1}$

(e) $\dfrac{x}{\sqrt{x^2+1}}$

11. $\arccos\left(\cos\left(\frac{4\pi}{3}\right)\right) =$ $\arccos\left(-\frac{1}{2}\right) = \pi - \arccos\left(\frac{1}{2}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$



(a) $\frac{4\pi}{3}$

(b) $\frac{2\pi}{3}$

(c) $\frac{\pi}{3}$

(d) $-\frac{\pi}{3}$

(e) $-\frac{4\pi}{3}$

12. $\sum_{i=1}^{100}(2^i - 2^{i+1}) =$

(a) $1-2^{101}$

(b) $1-2^{100}$

(c) $2-2^{100}$

(d) -2^{101}

(e) $2-2^{101}$

13. Find the equation of the tangent line to the graph of $f(x) = \ln(x^2)$ at $x = e$.

(a) $y - 2e = 2e(x - e)$

(b) $y - 2 = \frac{2}{e}(x - e)$

(c) $y - 2e = \frac{2}{e}(x - e)$

(d) $y - 2 = \frac{1}{e^2}(x - e)$

(e) $y - \frac{2}{e} = 2e(x - e)$

$f(x) = 2\ln x$, $f(e) = 2$

$f'(x) = \frac{2}{x}$, $f'(e) = \frac{2}{e}$

$y = f'(e)(x - e) + f(e)$

$y = \frac{2}{e}(x - e) + 2$

$y - 2 = \frac{2}{e}(x - e)$

$y = \frac{2x}{e} - 2 + 2$

$y = \frac{2x}{e}$

14. Solve for x: $\ln(x) + \ln(x+1) = \ln(x+4)$

(a) ~~$x = 2$ and $x = -2$~~

(b) $x = 3$

(c) $x = 3$ and $x = 2$

(d) $x = 4$

(e) $x = 2$

Domain: $\boxed{x > 0}$

$x + 1 > 0 \Rightarrow x > -1$

$x + 4 > 0 \Rightarrow x > -4$

$\ln x(x+1) = \ln(x+4)$

$x(x+1) = x + 4$

$x^2 + x - x - 4 = 0$

$x^2 - 4 = 0 \Rightarrow \boxed{x = 2}$ and $x = -2$

15. If $f'(x) = (x-2)^3(4-x)^5(x+1)^2$, where does $f(x)$ have a local minimum?

(a) $x = 4$

(b) $x = -1$

(c) $x = 4$ and $x = -1$

(d) $x = 2$

(e) $x = -1$ and $x = 2$

$f(x)$

$f'(x)$: $-$ | -1 | $-$ | 2 | $+$ | 4 | $-$

2: $\boxed{\text{local min}}$

4: local max

$f'(0) = (-2)^3(4^5)\,1 < 0$

$f'(1) = (-1)^3(3^5)2^2 < 0$

$f'(3) = 1^3 \cdot 1^5 \cdot 4^2 > 0$

$f'(5) = 3^3 \cdot (-1)^5 \cdot 6^2 < 0$

PART II: Work Out

Directions: Present your solutions in the space provided. *Show all your work* neatly and concisely and *Box your final answer*. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

16. Consider $f(x) = (x-2)e^{3x}$.

(i) (4 pts) Find the intervals where $f(x)$ is increasing and decreasing.

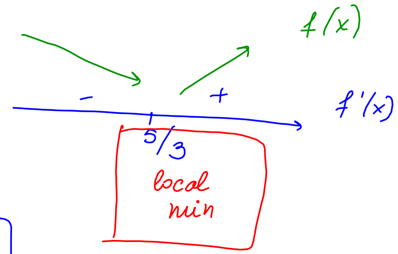
$f'(x) = e^{3x} + (x-2)\,3e^{3x}$

$= e^{3x}(1+3x-6)$

$f'(x) = \underbrace{e^{3x}}_{\text{always positive}}(3x-5) > 0$

$3x - 5 > 0$

$x > \frac{5}{3}$



$$\boxed{\begin{array}{l} f \text{ is increasing on } \left(\frac{5}{3}, \infty\right) \\ \quad\ \ \text{decreasing on } \left(-\infty, \frac{5}{3}\right) \end{array}}$$

(ii) (2 pts) Find the local maximum and local minimum of $f(x)$. If none, say *NONE*.

$f\left(\frac{5}{3}\right) = \left(\frac{5}{3} - 2\right)e^{3\frac{5}{3}} = -\frac{1}{3}e^5$

$$\boxed{\begin{array}{c} \text{local min @ } \left(\frac{5}{3}, -\frac{1}{3}e^5\right) \\ \text{no local max} \end{array}}$$

(iii) (4 pts) Find the intervals where $f(x)$ is concave up and concave down.

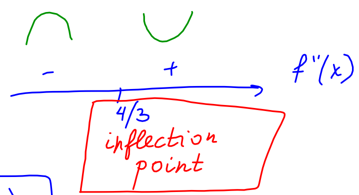
$f'(x) = (3x-5)e^{3x}$

$f''(x) = 3e^{3x} + (3x-5)(3)e^{3x}$

$= 3e^{3x}(1+3x-5)$

$= 3e^{3x}(3x-4) > 0$

$x > 4/3$



$$\boxed{\begin{array}{l} f \text{ is CU on } \left(\frac{4}{3}, \infty\right) \\ f \text{ is CD on } \left(-\infty, \frac{4}{3}\right) \end{array}}$$

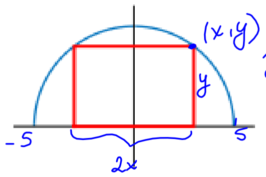
(iv) (2 pts) Find the inflection point(s) of $f(x)$. If none, say *NONE*.

inflection point @ $x = 4/3$

$f(4/3) = \left(\frac{4}{3} - 2\right)e^{3 \cdot 4/3} = -\frac{2}{3}e^4$

$$\boxed{\text{inflection point @ } \left(\frac{4}{3}, -\frac{2}{3}e^4\right)}$$

17. (10 pts) A rectangle is bounded by the x-axis and the semicircle $f(x) = \sqrt{25 - x^2}$ as shown below. What length and width should the rectangle have so that its area is a maximum?



$y = \sqrt{25-x^2}$

$A = (2x)y = 2x\sqrt{25-x^2}$

$(25-x^2)'$

$A'(x) = 2\sqrt{25-x^2} + 2x \frac{1}{2}(25-x^2)^{-1/2}(-2x)$

$= \dfrac{2\sqrt{25-x^2} - \dfrac{2x^2}{\sqrt{25-x^2}}}{2} = \dfrac{0}{2}$

$\left(\sqrt{25-x^2} = \dfrac{x^2}{\sqrt{25-x^2}}\right)\left(\sqrt{25-x^2}\right)$

$25 - x^2 = x^2$

$25 = 2x^2 \Rightarrow x^2 = \frac{25}{2}, \quad \boxed{x = \frac{5}{\sqrt{2}}}$

$y = \sqrt{25 - \frac{25}{2}} = \frac{5}{\sqrt{2}}$

Dimensions: $\boxed{\frac{10}{\sqrt{2}} \times \frac{5}{\sqrt{2}}}$ $(2x \times y)$

Show that $A''\left(\frac{5}{\sqrt{2}}\right) < 0$

$-5 \le x \le 5$

$A(-5) = 0$

$A(5) = 0$

$A\left(\frac{5}{\sqrt{2}}\right) = 2 \cdot \frac{5}{\sqrt{2}}\sqrt{25 - \frac{5^2}{2}}$

$= 2 \cdot \frac{25}{2} = \boxed{25 \text{ abs max}}$

$A''(x) = 2\frac{1}{2}(25-x^2)^{-1/2}(25-x^2)' - \dfrac{4x\sqrt{25-x^2} - 2x^2\frac{1}{2}(25-x^2)^{-1/2}(-2x)}{25-x^2}$

$= \dfrac{(-2x)}{\sqrt{25-x^2}} - \dfrac{4x\sqrt{25-x^2} + 2x^3(25-x^2)^{-1/2}}{25-x^2}$

$A''\left(\frac{5}{\sqrt{2}}\right) = \dfrac{-2 \cdot \frac{5}{\sqrt{2}}}{\sqrt{25 - \frac{25}{2}}} - \dfrac{4\frac{5}{\sqrt{2}}\sqrt{25 - \frac{25}{2}} + 2\left(\frac{5}{\sqrt{2}}\right)^3\left(25 - \frac{25}{2}\right)^{-1/2}}{25 - \frac{25}{2}}$

$= -2 - \dfrac{4 \cdot \frac{25}{2} + 2 \cdot \frac{125}{2\sqrt{2}}\left(\frac{25}{2}\right)^{-1/2}}{\frac{25}{2}}$

$= -2 - \dfrac{\frac{100}{2} + \cancel{2} \cdot \frac{125}{\cancel{2\sqrt{2}}} \cdot \frac{\cancel{\sqrt{2}}}{5}}{\frac{25}{2}}$

$= -2 - \dfrac{\frac{100}{2} + \frac{125}{5}}{\frac{25}{2}} < 0$

18. (11 pts) Find $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{3x}$.

$|1^\infty| = \lim_{x\to\infty} e^{3x\ln(1+2/x)} = e^{\lim_{x\to\infty} 3x\ln(1+2/x)}$

$\lim_{x\to\infty} 3x\ln\left(1+\frac{2}{x}\right) = |\infty\cdot 0|$

$= 3\lim_{x\to\infty}\frac{\ln(1+2/x)}{1/x} = \frac{0}{0} = 3\lim_{x\to\infty}\frac{\frac{1}{1+2/x}\left(-\frac{2}{\cancel{x^2}}\right)}{-1/\cancel{x^2}}$

$= 3\lim_{x\to\infty}\frac{-\frac{2}{1+\cancel{2/x}\to 0}}{-1} = 3(2) = 6$

$\boxed{e^6}$

19. (11 pts) Given that $\mathbf{r}''(t) = \langle e^t + t, \cos t - 1\rangle$, $\mathbf{r}'(0) = \langle 1, -2\rangle$ and $\mathbf{r}(0) = \langle 4, 12\rangle$, find $\mathbf{r}(t)$.

$\vec{r}'(t) = \langle e^t + \frac{t^2}{2} + C_1, \sin t - t + C_2\rangle$, $\vec{r}'(0) = \langle 1 + C_1, C_2\rangle = \langle 1, -2\rangle$

$1 + C_1 = 1 \Rightarrow C_1 = 0$

$C_2 = -2$

$\vec{r}'(t) = \langle e^t + \frac{t^2}{2}, \sin t - t - 2\rangle$

$\vec{r}(t) = \langle e^t + \frac{t^3}{6} + C_3, -\cos t - \frac{t^2}{2} - 2t + C_4\rangle$

$\vec{r}(0) = \langle 1 + C_3, -1 + C_4\rangle = \langle 4, 12\rangle$

$1 + C_3 = 4 \Rightarrow C_3 = 3$

$-1 + C_4 = 12 \Rightarrow C_4 = 13$

$$\boxed{\vec{r}(t) = \langle e^t + \frac{t^3}{6} + 3, -\cos t - \frac{t^2}{2} - 2t + 13\rangle}$$

20. Find the derivative of $f(x)$. Do not simplify.

(i) (4 pts) $f(x) = 4^{x \tan x}$

$$f'(x) = 4^{x\tan x} (x \tan x)' \ln 4$$

$$= \boxed{4^{x\tan x} (\tan x + x \sec^2 x) \ln 4}$$

(ii) (4 pts) $f(x) = \arccos(x^2)$

$$f'(x) = -\frac{1}{\sqrt{1-(x^2)^2}} (x^2)'$$

$$= \boxed{-\frac{2x}{\sqrt{1-x^4}}}$$

$$(\log_a x)' = \frac{1}{x \ln a}$$

(iii) (3 pts) $f(x) = \log_3(\ln x)$

$$f'(x) = \frac{1}{\ln x \cdot \ln 3} (\ln x)'$$

$$= \boxed{\frac{1}{x \ln x \cdot \ln 3}}$$