Tuesday, Nov. 29, 7:30-9:30 PM in ARCC 105.
Green long scantron.
over 4.2-4.6, 4.8, 5.1-5.3, 5.5, 5.7

1. If $g(x)$ is the inverse of $f(x) = \sqrt{-x^3 - 3x + 2}$, what is $g'(4)$?

(a) $-\frac{15}{8}$

(b) $\frac{15}{8}$

(c) $\frac{8}{15}$

(d) $-\frac{8}{15}$

(e) None of these

$g'(4) = \frac{1}{f'(g(4))}$,

$g(4) = x \Rightarrow f(x) = 4$

$-x^3 - 3x + 2 = 4^2 = 16$

$x = -2$

$8 + 6 + 2 = 16 \Rightarrow g(4) = -2.$

$f'(x) = \frac{1}{2}(-x^3 - 3x + 2)^{-1/2}(-3x^2 - 3) = -\frac{3}{2}\frac{x^2+1}{\sqrt{-x^3 - 3x + 2}}$

$f'(-2) = -\frac{3}{2}\frac{(-2)^2+1}{\sqrt{-(-2)^3 - 3(-2) + 2}} = -\frac{3}{2}\frac{5}{4} = -\frac{15}{8}$

$g'(4) = \frac{1}{f'(-2)} = \boxed{-\frac{8}{15}}$

2. Find $f'(e)$ if $f(x) = \ln(x + \ln x)$.

(a) $\frac{1}{e+1}$

(b) $\frac{1}{e}$

(c) $\frac{1}{1+e^2}$

(d) $\frac{1}{e^2+e}$

(e) $\frac{e+1}{e^2}$

$f'(x) = \frac{1}{x + \ln x}(x + \ln x)' = \frac{1}{x + \ln x}\left(1 + \frac{1}{x}\right) = \frac{1}{x + \ln x}\frac{x+1}{x}$

$f'(e) = \frac{1}{e + \ln e} \cdot \frac{e+1}{e} = \boxed{\frac{1}{e}}$

3. Which of the following expressions is equivalent to $\sin(\tan^{-1} x)$?

(a) $\frac{x}{\sqrt{1+x^2}}$

(b) $\frac{x}{\sqrt{1-x^2}}$

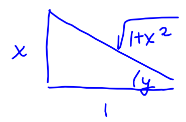
(c) $\frac{1}{\sqrt{1+x^2}}$

(d) $\frac{x}{1+x}$

(e) x

$\tan^{-1} x = y \Leftrightarrow \tan y = x$

$\sin y = \frac{x}{\sqrt{1+x^2}}$



4. If $f'(x) = \dfrac{(2x+1)^2}{x}$ and $f(2) = 0$, what is $f(1)$?

(a) $-9 - \ln 2$
(b) $2 - \ln 2$
(c) $-10 - \ln 2$
(d) $2 + \ln \frac{1}{2}$
(e) $-16 + \ln 2$

$f'(x) = (4x^2+4x+1)x^{-1} = 4x + 4 + \frac{1}{x}$

$f(x) = 4\frac{x^2}{2} + 4x + \ln|x| + C$

$= 2x^2 + 4x + \ln|x| + C$

$f(2) = 2(4) + 4(2) + \ln 2 + C = 0$

$C = -16 - \ln 2$

$f(x) = 2x^2 + 4x + \ln|x| - 16 - \ln 2$

$f(1) = 2 + 4 + \ln 1 - 16 - \ln 2$

$= \boxed{-10 - \ln 2}$

5. A bacteria culture starts with 4 million bacteria and the population triples every 30 minutes. The number of bacteria after 90 minutes is:

(a) 24 million
(b) 36 million
(c) 45 million
(d) 81 million
(e) 108 million

$p(60) = 3p(30)$
$= 36 \times 10^6$

$p(90) = 3p(60)$
$= 108 \times 10^6$

$p(0) = 4 \times 10^6$

$p(30) = 3p(0) = 12 \times 10^6$

$p(t) = p(0)e^{kt}$

$p(30) = 3p(0) = p(0)e^{30k}$

$3 = e^{30k}$

$30k = \ln 3$

$k = \frac{\ln 3}{30}$

$p(t) = p(0)e^{t\left(\frac{\ln 3}{30}\right)}$

$p(90) = p(0)e^{90\left(\frac{\ln 3}{30}\right)}$

$= p(0)e^{3\ln 3}$

$= p(0)(3^3) = 27(4 \times 10^6)$

$= 108 \times 10^6$

6. Consider the function $f(x) = 3x^4 - 8x^3 + 5$. At what value(s) of x does $f(x)$ have a local maximum?

(a) $x = 0$ and $x = 2$
(b) $x = 0$
(c) $x = 2$
(d) $x = \frac{4}{3}$
(e) f has no local maxima.

$f'(x) = 12x^3 - 24x^2 = 0$

$12x^2(x-2) = 0$

$x=3:\ 12(3^2)(3-2) > 0$

$x=1:\ 12(1^2)(1-2) < 0$

$x=-1:\ 12(-1)^2)(-1-2) < 0$

$-$ $-$ $+$

0

$\boxed{2 \text{ local min}}$

7. Find a number c that satisfies the conclusion of the Mean Value Theorem for $f(x) = x^3 + x - 1$ on the interval $[0, 2]$.

(a) 1
(b) $\frac{2}{\sqrt{3}}$
(c) $\frac{1}{3}$
(d) $\sqrt{3}$
(e) $\frac{3}{2}$

$f'(c) = \frac{f(b)-f(a)}{b-a}$,

$f(2) = 8+2-1 = 9$

$f(0) = -1$

$f'(x) = 3x^2+1$

Find $0<c<2$, such that $3c^2+1 = \frac{9-(-1)}{2-0} = 5$

$3c^2+1 = 5$

$3c^2 = 4$

$c^2 = \frac{4}{3}$, $c = \boxed{\frac{2}{\sqrt{3}}}$, $c = -\frac{2}{\sqrt{3}}$

8. Find the equation of the tangent line of $f(x) = (\ln x)^3$ at $x = e$.

(a) $y = \frac{5}{e}x - 4$
(b) $y = \frac{1}{e}x$
(c) $y = 3x - 3e + 1$
(d) $y = \frac{3}{e}x - 2$
(e) None of these

$f'(x) = 3(\ln x)^2 \frac{1}{x}$

$f'(e) = 3(\ln e)^2 \frac{1}{e} = \frac{3}{e}$

$f(e) = (\ln e)^3 = 1$

$y = \frac{3}{e}(x-e)+1$

$y = \frac{3}{e}x - 3 + 1$, $\boxed{y = \frac{3}{e}x - 2}$

9. Find $\lim_{x\to 0} \cos x^{\frac{1}{x^2}}$. $= \lim_{x\to 0} e^{\frac{1}{x^2}\ln(\cos x)} = e^{\lim_{x\to 0}\frac{\ln(\cos x)}{x^2}}$

$$\lim_{x\to 0}\frac{\ln(\cos x)}{x^2} \quad \left/\frac{0}{0}\right/ = \lim_{x\to 0}\frac{\frac{1}{\cos x}(-\sin x)}{2x}$$

$$= -\frac{1}{2}\lim_{x\to 0}\frac{\sin x}{(\cancel{\cos x})(x)} = -\frac{1}{2}\lim_{x\to 0}\cancel{\frac{\sin x}{x}}^{\,1} = -\frac{1}{2}$$

$$\boxed{e^{-1/2} = \frac{1}{\sqrt{e}}}$$

(a) 1
(b) 0
(c) $\frac{1}{\sqrt{e}}$
(d) ∞
(e) None of these

10. Find the **SUM** of the solutions. If there is only one answer, give it.

$$\log_2(x+4) + \log_2(x-3) = 3$$

$\log_2 (x+4)(x-3) = 3$

$(x+4)(x-3) = 2^3$

$x^2 + 4x - 3x - 12 = 8$

$x^2 + x - 12 - 8 = 0$

$x^2 + x - 20 = 0$

$(x+5)(x-4) = 0$

$x_1 = -5,\ \boxed{x_2 = 4}$

Domain: $x+4>0;\ x>-4$
$x-3>0;\ x>3$

$x>3$

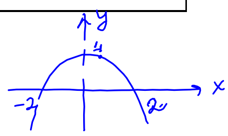
(a) -1
(b) 4
(c) 0
(d) -5
(e) No solution

11. What is the domain of $f(x) = \dfrac{\ln(4-x^2)}{\sqrt{x}}$?

ln: $4-x^2>0 \Rightarrow -2<x<2$

$\sqrt{x}$: $x>0$

$\boxed{(0,2)}$



(a) $[0,2)$
(b) $(2,\infty)$
(c) $(-2,0)$
(d) $(0,2)$
(e) $(-\infty,-2)\cup(2,\infty)$

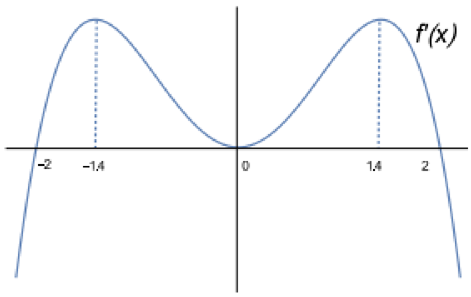
12. Evaluate $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$. $= \lim_{x\to 1} \frac{x-1-\ln x}{(x-1)\ln x} \left|\frac{0}{0}\right| = \lim_{x\to 1} \frac{1-\frac{1}{x}}{\ln x + \frac{x-1}{x}}$

(a) $\frac{1}{2}$
(b) 0
(c) 1
(d) -1
(e) $-\infty$

$= \lim_{x\to 1} \frac{\frac{x-1}{x}}{\frac{x\ln x + x - 1}{x}} = \lim_{x\to 1} \frac{x-1}{x\ln x + x - 1}, \left|\frac{0}{0}\right|$

$= \lim_{x\to 1} \frac{1}{\ln x + x\frac{1}{x} + 1} = \boxed{\frac{1}{2}}$

13. If the graph below is that the derivative, $f'(x)$, of a continuous function $f(x)$, on which interval(s) is $f(x)$ concave up?



f is CU when f' is increasing

$(-\infty, -1.4) \cup (0, 1.4)$

(a) $(-1.4, 1.4)$
(b) $(-\infty,-2)\cup(2,\infty)$
(c) ~~$(-1.4, 0)\cup(1.4,\infty)$~~
(d) $(-2, 2)$
(e) $(-\infty,-1.4)\cup(0,1.4)$

14. Find the values of the absolute minimum and maximum for $f(x) = 1 + x + \frac{4}{x}$ on the interval $\underbrace{[1, 3]}$.

(a) -3 and 6

(b) 5 and $\frac{16}{3}$

(c) -3 and 6

(d) 5 and 6

(e) None of these

$f'(x) = 1 - \frac{4}{x^2} = 0$

$\frac{x^2-4}{x^2} = 0 \quad , x \neq 0$

$x^2 - 4 = 0$

$x = 2, \quad \underbrace{x = -2}$ out the interval

$f(1) = 1+1+4 = 6 = \boxed{\frac{18}{3} \text{ abs max}}$

$f(2) = 2+1+2 = 5 = \boxed{\frac{15}{3} \text{ abs min}}$

$f(3) = 1+3+\frac{4}{3} = \frac{16}{3}$

15. Find the inverse function of $f(x) = \ln(1 - 2x)$.

(a) $f^{-1}(x) = \frac{1-e^x}{2}$

(b) $f^{-1}(x) = \frac{1+e^x}{2}$

(c) $f^{-1}(x) = -\ln(1 - 2x)$

(d) $f^{-1}(x) = \frac{1}{\ln(1-2x)}$

(e) None of these

$y = \ln(1-2x)$

$e^y = 1 - 2x$

$2x = 1 - e^y$

$x = \frac{1-e^y}{2} = f^{-1}(y) \quad , \quad f^{-1}(x) = \frac{1-e^x}{2}$

PART II: Work Out

Directions: Present your solutions in the space provided. *Show all your work* neatly and concisely and *Box your final answer*. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

16. (8 pts) Find $f'(x)$. Do not simplify.

(a) $f(x) = \arctan(\sqrt{x^2+1}) - \frac{1}{2}\ln(1+x^2) + e^{x^2+3x}$

$$f'(x) = \frac{1}{1+(\sqrt{x^2+1})^2} \underbrace{\frac{1}{2}(x^2+1)^{-1/2}(2x)}_{(\sqrt{x^2+1})'} - \frac{1}{2}\frac{1}{1+x^2}(2x) + e^{x^2+3x}(2x+3)$$

(b) $f(x) = (2 + \cos x)^{\sin x}$

$$\ln f = \sin x \ln(2+\cos x)$$

$$\frac{f'}{f} = \cos x \ln(2+\cos x) + \sin x \frac{1}{2+\cos x} (2+\cos x)'$$

$$= \cos x \ln(2+\cos x) - \frac{\sin^2 x}{2+\cos x}$$

$$f' = f\left[\cos x \ln(2+\cos x) - \frac{\sin^2 x}{2+\cos x}\right]$$

$$f' = \boxed{(2+\cos x)^{\sin x}\left[\cos x \ln(2+\cos x) - \frac{\sin^2 x}{2+\cos x}\right]}$$

17. (8 pts) If $f''(x) = 6x + \cos x$ and $f(0) = 0$, and $f(2\pi) = 8\pi^3$, find $f(x)$.

$f'(x) = 6\frac{x^2}{2} + \sin x + C_1 = 3x^2 + \sin x + C_1$

$f(x) = 3\frac{x^3}{3} - \cos x + C_1 x + C_2$

$f(x) = x^3 - \cos x + C_1 x + C_2$

$f(0) = 0^3 - \cos 0 + C_1(0) + C_2 = 0$

$-1 + C_2 = 0 \Rightarrow C_2 = 1$

$f(x) = x^3 - \cos x + C_1 x + 1$

$f(2\pi) = (2\pi)^3 - \cos(2\pi) + 2\pi C_1 + 1 = 8\pi^3$

$8\pi^3 - 1 + 2\pi C_1 + 1 = 8\pi^3$

$2\pi C_1 = 0 \Rightarrow C_1 = 0$

$\boxed{f(x) = x^3 - \cos x + 1}$

18. (6 pts) Newton's Law of cooling states the rate of cooling of an object is proportional to the temperature difference between the object and the temperature of the objects surroundings. A thermometer is taken from a room where the temperature is $25°C$, to the outdoors where the temperature is $-5°C$. After one minute the thermometer reads $5°C$. What will the termometer read after one more minute?

$T(t)$

$$\frac{dT}{dt} = k(T-(-5))$$

$$\frac{dT}{dt} = k(T+5) \quad ; \quad u(t) = T(t)+5$$

$$T(t) = u(t) - 5$$

$$\frac{dT}{dt} = \frac{du}{dt}, \quad u(0) = 25+5 = 30$$

$$\frac{du}{dt} = ku, \quad u(0) = 30$$

$$u(t) = u(0)e^{kt}$$

$$T(t)+5 = u(t) = 30e^{kt}$$

$$T(t) = -5+30e^{kt}$$

$$T(1) = -5+30e^{k} = 5$$

$$30e^{k} = 10$$

$$e^{k} = 1/3$$

$$k = \ln\frac{1}{3} = -\ln 3$$

$$T(t) = -5+30e^{(-\ln 3)t} = -5+30(3^{-t})$$

$$T(2) = -5+\frac{30}{9} = \boxed{-\frac{5}{9}\ °C}$$

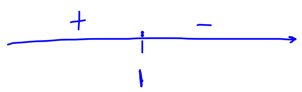
19. (10 pts) Consider the function $f(x) = xe^{-x}$.

(a) Find $\lim_{x\to\infty} f(x)$. $= \lim_{x\to\infty} xe^{-x} \; |0\cdot\infty| = \lim_{x\to\infty} \frac{x}{e^x} \; \left|\frac{\infty}{\infty}\right|$

$$= \lim_{x\to\infty} \frac{1}{e^x} = \boxed{0}$$

(b) Find the interval(s) where $f(x)$ is increasing or decreasing.

$$f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x} = 0, \quad x = 1$$



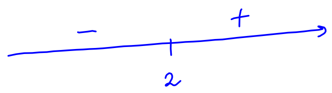
f is increasing on $(-\infty, 1)$
decreasing on $(1, \infty)$

(c) Find the local extrema of $f(x)$.

local max @ $(1, e^{-1})$

(d) Find the interval(s) of concavity of $f(x)$.

$$f''(x) = -e^{-x} + (1-x)(-e^{-x}) = -e^{-x}(2-x) = 0, \quad x = 2$$



f is CU on $(2, \infty)$
f is CD on $(-\infty, 2)$

(e) Find the point(s) of inflection for $f(x)$.

$(2, 2e^{-2})$ – inflection point.

20. (8 pts) A box with a square base and open top must have a volume of $4000cm^3$. Find the dimensions of the box that minimize the amount of material used.