- HW over 3.11, 4.1 is due 10/26, 11:55PM
- HW over 4.2, 4.3 is due 11/2, 11:55 PM
- Test 2 — 7:30-9:30 PM in ARCC 105 10/27.
  over 3.2, 3.4 – 3.11, 4.1

1. For which value of r does $y = e^{rx}$ satisfy the equation $y'' - 2y' + y = 0$?

(a) 2
(b) 1
(c) 0
(d) -1
(e) -2

$y' = e^{rx}(rx)'$
$= re^{rx}$

$y'' = (re^{rx})'$
$= r(e^{rx})'$
$= r(re^{rx})$
$= r^2e^{rx}$

$$\frac{\underbrace{r^2e^{rx}}_{y''} - 2\underbrace{re^{rx}}_{y'} + \underbrace{e^{rx}}_{y} = 0}{e^{rx}} \quad \overline{e^{rx}}$$

$r^2 - 2r + 1 = 0$
$(r-1)^2 = 0 \Rightarrow \boxed{r = 1}$

2. $\displaystyle\lim_{x\to-\infty} \frac{2e^{-3x} - 3e^{3x}}{4e^{-3x} + 2e^{3x}} =$

(a) ∞
(b) $-\frac{3}{2}$
(c) $-\frac{1}{6}$
(d) $\frac{1}{2}$
(e) $-\infty$

$\lim_{x\to-\infty} e^{-3x} = \infty$

$\lim_{x\to-\infty} e^{3x} = 0$

$e^{3x+3x} = e^{6x}$

$$\lim_{x\to-\infty} \frac{\frac{2}{e^{3x}} - 3e^{3x}}{\frac{4}{e^{3x}} + 2e^{3x}} = \lim_{x\to-\infty} \frac{\frac{2 - 3e^{3x}e^{3x}}{\cancel{e^{3x}}}}{\frac{4 + 2e^{3x}e^{3x}}{\cancel{e^{3x}}}} = \lim_{x\to-\infty} \frac{2 - 3\cancel{e^{6x}}^{\,0}}{4 + 2\cancel{e^{6x}}^{\,0}}$$

$= \frac{2}{4} = \frac{1}{2}$

3. A particle moves according to the equation of motion $s(t) = t^2 - 2t + 3$ where $s(t)$ is measured in feet and t is measured in seconds. Find the total traveled distance in the first 3 seconds.

(a) 2 feet
(b) 3 feet
(c) 5 feet
(d) 6 feet
(e) 11 feet

$V(t) = 2t - 2 > 0 \Rightarrow t > 1$

$t=1$ $\quad t=0$ $\quad t=3$

$|s(1) - s(0)| = |\underbrace{1-2+3}_{s(1)} - 3| = |-1| = 1$

$s(3) - s(1) = \underbrace{3^2 - 2(3) + 3}_{s(3)} - (1-2+3) = 9 - 6 + 1 = 4$

total distance $= 1 + 4 = 5$

4. The vector function $\mathbf{r}(t) = \langle t + e^{4t}, -t\cos(2t)\rangle$, $0 \le t \le 2\pi$, represents the position of a particle at time t. Find the acceleration vector of the object at $t = \frac{\pi}{4}$.

(a) $\langle 1 + 4e^{\pi}, -1\rangle$

(b) $\langle 4e^{\pi}, \frac{\pi}{2}\rangle$

(c) $\langle 1 + 16e^{\pi}, \pi\rangle$

(d) $\langle 16e^{\pi}, 4\rangle$

(e) $\langle 16e^{\pi}, 4 + \pi\rangle$

$\vec{a}(t) = \vec{r}''(t) = \vec{v}'(t)$

$\vec{v}(t) = \vec{r}'(t) = \langle 1 + 4e^{4t}, -\cos 2t - t(-\sin 2t)(2t)' \rangle$

$= \langle 1 + 4e^{4t}, -\cos 2t + 2t \sin 2t \rangle$

$\vec{a}(t) = \vec{v}'(t) = \langle 4e^{4t}(4t)', \sin(2t)(2t)' + 2\sin 2t + 2t\cos 2t(2t)' \rangle$

$= \langle 16e^{4t}, 4\sin 2t + 4t\cos 2t \rangle$

$\vec{a}\left(\frac{\pi}{4}\right) = \langle 16e^{4\frac{\pi}{4}}, 4\sin\cancel{\frac{\pi}{2}}^{1} + 4\frac{\pi}{4}\cos\cancel{\frac{\pi}{2}}^{0} \rangle$

$= \langle 16e^{\pi}, 4 \rangle$

5. Find $\lim_{x\to 0} \frac{\sin^2(3x)}{x^2 \cancel{\cos x}}$. $= \lim_{x\to 0} \frac{\sin^2(3x)}{x^2} = \lim_{x\to 0}\left(\frac{3\sin 3x}{3x}\right)^2 = 9\lim_{x\to 0}\cancel{\left(\frac{\sin 3x}{3x}\right)^2}^{1}$

$\lim_{x\to 0} \cos x = 1$

$= \boxed{9}$

(a) 3

(b) 4

(c) 6

(d) 9

(e) The limit does not exist.

6. Find the equation of the tangent line to the curve $\frac{d}{dx}(2x^2y - 3y^2) = \frac{d}{dx}(11)$ at the point $(2, -1)$.

(a) $y = \frac{4}{7}x - \frac{1}{7}$

(b) $y = -\frac{4}{7}x + \frac{1}{7}$

(c) $y = -\frac{4}{7}x - \frac{1}{7}$

(d) $y = -\frac{4}{7}x + \frac{15}{7}$

(e) $y = \frac{4}{7}x - \frac{15}{7}$

$4xy + 2x^2y' - 6y\,y' = 0$

$y'(2x^2 - 6y) = -4xy$

$y' = -\frac{4xy}{2x^2 - 6y} = -\frac{2xy}{x^2 - 3y}$

Plug $x = 2$, $y = -1$

$y'(2,-1) = -\frac{2(2)(-1)}{2^2 - 3(-1)} = -\frac{-4}{7} = \frac{4}{7}$

$y = \frac{4}{7}(x-2) - 1$

$\frac{4}{7}x - \frac{8}{7} - 1$

$\frac{4}{7}x - \frac{15}{7}$

(7-8) Suppose f and g are differentiable functions which satisfy the following condition.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-1	-1	3	5	0
1	-1	2	-1	2

7. Let $u(x) = f(x) \cdot g(x)$. Find $u'(1)$.

$u'(x) = f'(x)g(x) + f(x)g'(x)$
$u'(1) = f'(1)g(1) + f(1)g'(1)$
$= (2)(-1) + (-1)(2)$
$= -4$

   (a) 0
   (b) -1
   (c) -3
   (d) -4
   (e) It is impossible to determine the value from the information.

8. Let $v(x) = \dfrac{f(g(x))}{x^2}$. Find $v'(1)$.

$$v'(x) = \frac{[f(g(x))]'(x^2) - 2x\, f(g(x))}{x^4}$$
$$= \frac{f'(g(x))\, g'(x)(x^2) - 2x\, f(g(x))}{x^4}$$
$$v'(1) = \frac{f'(g(1))\, g'(1) - 2f(g(1))}{1} = f'(-1)g'(1) - 2f(-1)$$
$$= 3(2) - 2(-1)$$
$$= 6 + 2 = 8$$

   (a) 0
   (b) 2
   (c) 4
   (d) 6
   (e) 8

9. Find the quadratic approximation for $f(x) = e^{x^2}$ at $x = 0$.

$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2, \; a = 0$

$f(x) = e^{x^2}$	$f(0) = 1$
$f'(x) = e^{x^2}(x^2)' = 2xe^{x^2}$	$f'(0) = 0$
$f''(x) = 2e^{x^2} + 2x(e^{x^2})' = 2e^{x^2} + 2x(2xe^{x^2}) = 2e^{x^2} + 4x^2e^{x^2}$	$f''(0) = 2$

$f(x) \approx 1 + \frac{2}{2}x^2$
$= 1 + x^2$

   (a) $1 + x^2$
   (b) $1 + x + x^2$
   (c) $1 + 2x^2$
   (d) $1 + x + 2x^2$
   (e) 1

10. Find the value of x where the tangent line to the graph of $f(x) = \frac{x}{\sqrt{x}}$ $= x^{1/2}$ is parallel to the line $x - 4y = 10$.

(a) $\frac{1}{4}$
(b) $\frac{1}{2}$
(c) 1
(d) 2
(e) 4

$f'(x) = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}} = \frac{1}{4}$

$2\sqrt{x} = 4$

$\sqrt{x} = 2$

$x = 4$

$4y = x - 10$

$y = \frac{1}{4}x - \frac{10}{4}$

$m = \frac{1}{4}$

11. At what point does the curve parametrized by $x = t^2 - 6t + 5$, $y = t^2 + 4t + 3$ have a vertical tangent?

(a) $(-11, 35)$
(b) $(-4, 24)$
(c) $(0, 8)$
(d) $(5, 1)$
(e) $(21, -1)$

$x'(t) = 0$ – vertical tangent

$x'(t) = 2t - 6 = 0 \Rightarrow t = 3$

$x(3) = 9 - 18 + 5 = -4$

$y(3) = 9 + 12 + 3 = 24$

12. Given the curve parametrized by $x = \overbrace{\sqrt{2t}}^{\sqrt{2}\cdot t^{1/2}}$, $y = \sin\frac{\pi t}{2}$, find the slope of the line tangent to the curve at the point $(2, 0)$.

(a) $-\pi$
(b) $-\frac{1}{\pi}$
(c) 2
(d) $-\frac{1}{2}$
(e) $\frac{1}{2}$

$\sqrt{2t} = 2$

$2t = 4$

$\boxed{t = 2.}$

$y(2) = \sin\frac{2\pi}{2} = \sin\pi = 0.$

$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\cos\frac{\pi t}{2}\left(\frac{\pi t}{2}\right)'}{\sqrt{2}\cdot\frac{1}{2}t^{-1/2}} = \frac{\frac{\pi}{2}t^{1/2}\cos\frac{\pi t}{2}}{\frac{\sqrt{2}}{2}} = \frac{\pi}{2}t^{1/2}\cos\frac{\pi t}{2}\cdot\frac{2}{\sqrt{2}}$

$= \frac{\pi}{\sqrt{2}} t^{1/2}\cos\frac{\pi t}{2}$

$\left.\frac{dy}{dx}\right|_{t=2} = \frac{\pi}{\sqrt{2}}\sqrt{2}\cos\frac{2\pi}{2} = -\pi$

13. Find $f^{(2016)}(x)$ for $f(x) = \frac{-1}{(x+1)^2}$, $x \neq -1$.

(a) $f^{(2016)}(x) = -\frac{2016!}{(x+1)^{2017}}$

(b) $f^{(2016)}(x) = \frac{2016!}{(x+1)^{2017}}$

(c) $f^{(2016)}(x) = -\frac{2017!}{(x+1)^{2017}}$

(d) $f^{(2016)}(x) = \frac{2017!}{(x+1)^{2018}}$

(e) $f^{(2016)}(x) = -\frac{2017!}{(x+1)^{2018}}$

$$f(x) = -(x+1)^{-2}$$
$$f'(x) = -1(-2)(x+1)^{-3}$$
$$= (-1)^2(2)(x+1)^{-3}$$
$$f''(x) = (-1)^2\, 2(-3)(x+1)^{-4}$$
$$= (-1)^3 (2)(3)\,(x+1)^{-4}$$
$$f^{(n)}(x) = (-1)^{n+1}(n+1)!\,(x+1)^{-2-n}$$
$$f^{(2016)}(x) = (-1)^{2017}(2017)!\,(x+1)^{-2018}$$
$$= -\frac{2017!}{(x+1)^{2018}}$$

14. Find $f'(x)$ if $f(x) = \sqrt{\cos(\sin x)}$.

(a) $\frac{-\sin(\sin x)\cdot\cos x}{2\sqrt{\cos(\sin x)}}$

(b) $\frac{\cos x}{2\sqrt{\cos(\sin x)}}$

(c) $\frac{-\sin x\cdot\cos x}{2\sqrt{\cos(\sin x)}}$

(d) $\frac{-\sin^2 x\cdot\cos x}{2\sqrt{\cos(\sin x)}}$

(e) $\frac{\sin(\sin x)\cdot\cos x\cdot\sin x}{2\sqrt{\cos(\sin x)}}$

$$= [\cos(\sin x)]^{1/2}$$
$$f'(x) = \frac{1}{2}[\cos(\sin x)]^{-1/2}\,(\cos(\sin x))'$$
$$= \frac{1}{2}[\cos(\sin x)]^{-1/2}\,(-\sin(\sin x))\,(\sin x)'$$
$$= \frac{1}{2}[\cos(\sin x)]^{-1/2}\,(-\sin(\sin x)\cos x)$$
$$= -\frac{\cos x\sin(\sin x)}{2\sqrt{\cos(\sin x)}}$$

$$f(x) \approx f(3) + f'(3)(x-3) = \overbrace{f(3) - 3f'(3)}^{-2} + \overbrace{f'(3)}^{2}x$$

15. Suppose the linear approximation for the function $f(x)$ at $a = 3$ is given by $y = 2x - 2$. If $g(x) = \sqrt{f(x)}$, find the linear approximation for $g(x)$ at $a = 3$.

(a) $2 + \frac{1}{2}(x-3)$

(b) $2 + \sqrt{2}(x-3)$

(c) $4 + 2(x-3)$

(d) $4 + \frac{1}{2}(x-3)$

(e) None of these

$$\boxed{f'(3) = 2}$$
$$f(3) - 3f'(3) = -2$$
$$f(3) = -2 + 3f'(3)$$
$$= -2 + 3(2) = \boxed{4 = f(3)}$$

$$g(x) = \sqrt{f(x)} = [f(x)]^{1/2}$$
$$g'(x) = \frac{1}{2}[f(x)]^{-1/2}\,f'(x) = \frac{f'(x)}{2\sqrt{f(x)}}$$

$$g(3) = \sqrt{f(3)} = 2$$
$$g'(3) = \frac{2}{2\sqrt{4}} = \frac{1}{2}$$

16. (6 pts) Consider the parametric curve $x = \frac{1}{3}t^3 - 2t^2 + 3t - 5$ and $y = t^2 + 2t$. Find the equation of the tangent line at $t = 2$.

$$x(2) = \frac{1}{3}(8) - 2(4) + 3(2) - 5$$

$$= \frac{8}{3} - 8 + 6 - 5 = \frac{8}{3} - 7 = \frac{8-21}{3} = -\frac{13}{3}$$

$$y(2) = 4 + 4 = 8$$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{2t+2}{t^2-4t+3}$$

$$\left.\frac{dy}{dx}\right|_{t=2} = \frac{4+2}{4-8+3} = \frac{6}{-1} = -6$$

$$\boxed{y = -6\left(x + \frac{13}{3}\right) + 8}$$

17. (8 pts) Find $f'(x)$. Do not simplify.

(a) $f(x) = \dfrac{(4-x)^2}{\tan x}$

$$f'(x) = \frac{[(4-x)^2]'\tan x - (\tan x)'(4-x)^2}{\tan^2 x}$$

$$= \frac{2(4-x)(4-x)'\tan x - \sec^2 x\,(4-x)^2}{\tan^2 x}$$

$$= \boxed{\frac{-2(4-x)\tan x - \sec^2 x\,(4-x)^2}{\tan^2 x}}$$

(b) $g(x) = \sin^4\left(\pi^3 + \dfrac{1}{x^2}\right)$

$$g'(x) = 4\sin^3\left(\pi^3 + \frac{1}{x^2}\right)\left[\sin\left(\pi^3 + \frac{1}{x^2}\right)\right]'$$

$$= 4\sin^3\left(\pi^3 + \frac{1}{x^2}\right)\cos\left(\pi^3 + \frac{1}{x^2}\right)\left(\pi^3 + \frac{1}{x^2}\right)'$$

$$= \boxed{4\sin^3\left(\pi^3 + \frac{1}{x^2}\right)\cos\left(\pi^3 + \frac{1}{x^2}\right)(-2)x^{-3}}$$

18. (8 pts) Let $f(x) = \sqrt{x}$.

(a) Find the linear approximation for $f(x)$ at $x = 16$.

$f(x) = x^{1/2}$ | $f(16) = \sqrt{16} = 4$

$f'(x) = \frac{1}{2} x^{-1/2}$ | $f'(16) = \frac{1}{2\sqrt{16}} = \frac{1}{8}$

$$\sqrt{x} \approx 4 + \frac{1}{8}(x-16)$$

(b) Use the linear approximation above to approximate $\sqrt{16.03}$.

$$\sqrt{16.03} \approx 4 + \frac{1}{8}(16.03 - 16)$$

$$4 + \frac{0.03}{8} = \boxed{\frac{32.03}{8}}$$

19. (8 pts) Consider $f(x) = \begin{cases} ax^2 + x + 1 & \text{if } x \leq 1 \\ bx - 1 & \text{if } x > 1 \end{cases}$

(a) Find the value of a and b that make $f(x)$ differentiable everywhere.

(b) For the value of a and b found above, find $f'(x)$.

20. (10 pts) Water is leaking out of an inverted conical tank at a rate of $1m^3/min$. The tank has height $6m$ and the diameter at the top is $4m$. At what rate is the water level changing when the height of the water is $3m$?