Appendix D. **Trigonometry.**

Angles can be measured in degrees or radians. The angle given by a complete revolution contains 360^o, or 2π radians.

$$360^o = 2\pi \text{ rad}$$

$$1 \text{ rad} = \left(\frac{180}{\pi}\right)^o$$

$$1^o = \frac{\pi}{180} \text{ rad}$$

Example 1.

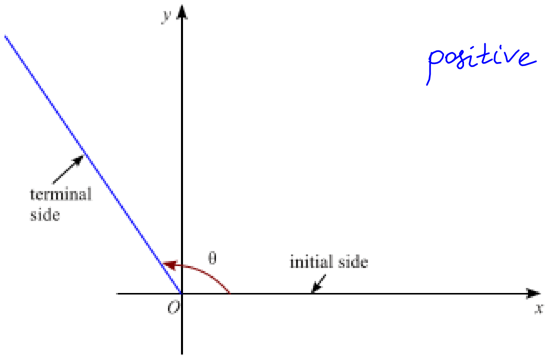
(a.) Convert 9^o to radians.

$$9^\circ = \frac{9\pi}{180}(\text{rad}) = \boxed{\frac{\pi}{20}} (\text{rad})$$

(b.) Convert $\frac{5\pi}{12}$ to degrees.

$$\frac{5\pi}{12} = \frac{5\pi}{12}\left(\frac{180}{\pi}\right)^\circ = \left(\frac{150}{2}\right)^\circ = \boxed{75^\circ}$$

The standard position of an angle occurs when we place its vertex at the origin of a coordinate system and its initial side on the positive x-axis. A positive angle is obtained by rotating the initial side counterclockwise until it coincides with the terminal side.

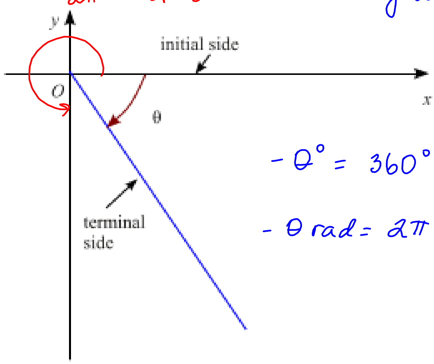


positive angle

Negative angles are obtained by clockwise rotation

$2\pi - \theta$ or $360° - \theta$

negative angle

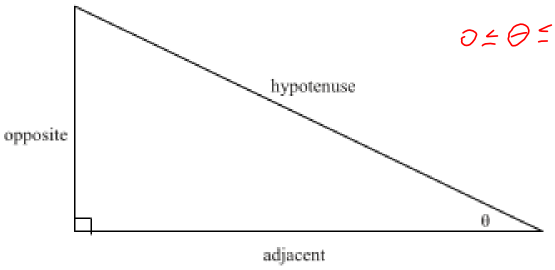


$-\theta° = 360° - \theta°$

$-\theta \text{ rad} = 2\pi - \theta$

The trigonometric functions.

$0 \le \theta \le \frac{\pi}{2}$

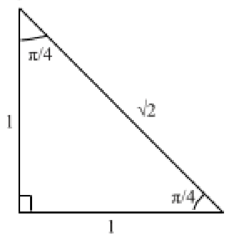


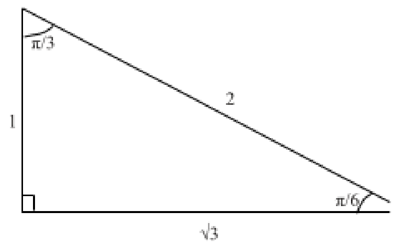
$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{1}{\tan\theta}$$

$$\frac{1}{\cos\theta} = \sec\theta = \frac{\text{hyp}}{\text{adj}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{1}{\sin\theta}$$

Special triangles:



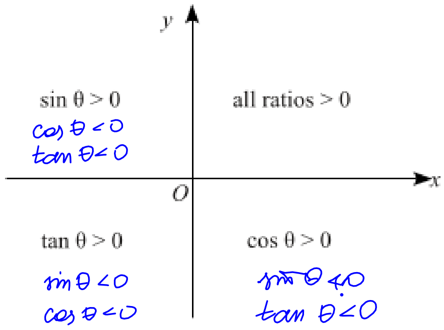


$$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad \sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\tan\frac{\pi}{4} = 1 \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \tan\frac{\pi}{3} = \sqrt{3}$$

Signs of the trigonometric functions:



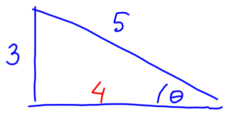
Example 2.
(a.) If $\sin\theta = \frac{3}{5}$ $(0 < \theta < \pi/2)$, find the remaining trigonometric rations.

1st quadrant

$\sin\theta = \frac{3}{5}$

$\cos\theta = \frac{4}{5}$

$\tan\theta = \frac{3}{4}$



$\sqrt{5^2 - 3^2} = \sqrt{25 - 9} = \sqrt{16} = 4$

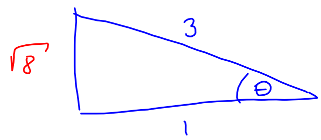
3rd quadrant, only $\tan\theta > 0$, thus, $\sin\theta < 0$

(b.) If $\cos\theta = -\frac{1}{3}$ $(\pi < \theta < 3\pi/2)$, find the remaining trigonometric rations.

$\cos\theta = -\frac{1}{3}$

$\sin\theta = -\frac{\sqrt{8}}{3}$

$\tan\theta = \sqrt{8}$



Trigonometric Identities:

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = sec^2\theta$$

$$\cot^2\theta + 1 = \csc^2\theta$$

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$

$$\sin(\theta + 2\pi) = \sin\theta \qquad \cos(\theta + 2\pi) = \cos\theta$$

$$\tan(\theta + \pi) = \tan\theta \qquad \cot(\theta + \pi) = \cot\theta$$

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\sin x \cos y = \frac{1}{2}[\sin(x + y) + \sin(x - y)]$$

$$\cos x \cos y = \frac{1}{2}[\cos(x + y) + \cos(x - y)]$$

$$\sin x \sin y = \frac{1}{2}[\cos(x - y) - \cos(x + y)]$$

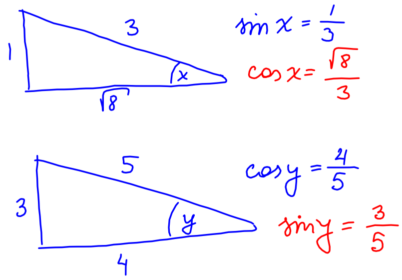
$\cos y = \frac{1}{\sec y} = \frac{4}{5}$

Example 4. If $\sin x = \frac{1}{3}$ and $\sec y = \frac{5}{4}$, where x and y lie between 0 and $\pi/2$, evaluate the expression

(a.) $\cos(x - y) = \cos x \cos y + \sin x \sin y$

$\cos x = \frac{\sqrt{8}}{3}$, $\cos y = 4/5$, $\sin x = 1/3$, $\sin y = \frac{3}{5}$

$= \frac{\sqrt{8}}{3} \cdot \frac{4}{5} + \frac{1}{3} \cdot \frac{3}{5} = \boxed{\frac{4\sqrt{8}+3}{15}}$



$\sin x = \frac{1}{3}$, $\cos x = \frac{\sqrt{8}}{3}$

$\cos y = \frac{4}{5}$, $\sin y = \frac{3}{5}$

4

(b.) $\sin 2y = 2 \sin y \cos y$

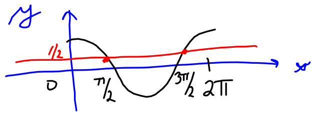
$= 2 \frac{4}{5} \cdot \frac{3}{5} = \boxed{\frac{24}{25}}$

Example 5. Find all values of x in the interval $[0, 2\pi]$ that satisfy the equation:

(a.) $2 \cos x - 1 = 0$

$2 \cos x = 1$

$\cos x = \frac{1}{2}$



$x = \boxed{\frac{\pi}{3}}, 2\pi - \frac{\pi}{3} = \boxed{\frac{5\pi}{3}}$

$x = \pm\frac{\pi}{3} + 2\pi n$

$n = 0$: $x = \boxed{\frac{\pi}{3}}$, $x = -\frac{\pi}{3}$

$n = 1$: $x = -\frac{\pi}{3} + 2\pi = \boxed{\frac{5\pi}{3}}$

(b.) $2 \cos x + \sin 2x = 0$

$\sin 2x = 2 \sin x \cos x$

$2 \cos x + 2 \sin x \cos x = 0$

$2 \cos x (1 + \sin x) = 0$

$\cos x = 0$ or $1 + \sin x = 0$

$x = \frac{\pi}{2}, x = \frac{3\pi}{2}$

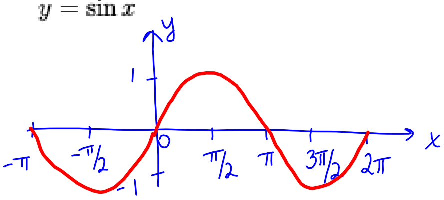
$\sin x = -1$

$x = \frac{3\pi}{2}$

$\boxed{x = \frac{\pi}{2}, \frac{3\pi}{2}}$

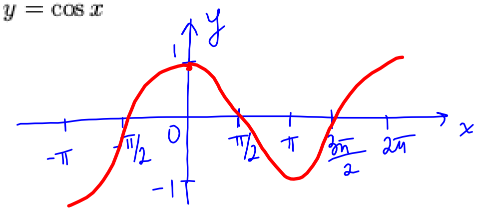
Graphs of the trigonometric functions.

$y = \sin x$



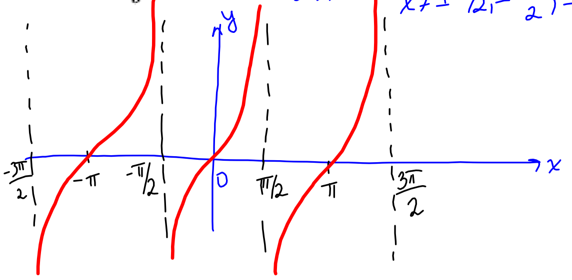
$-1 \le \sin x \le 1$

$y = \cos x$

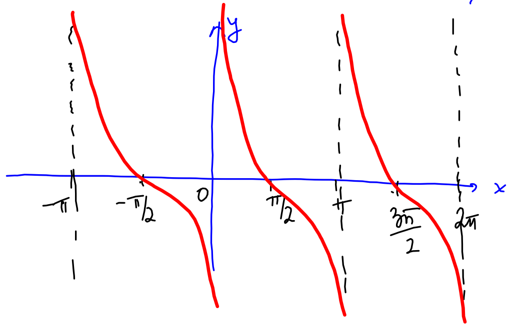


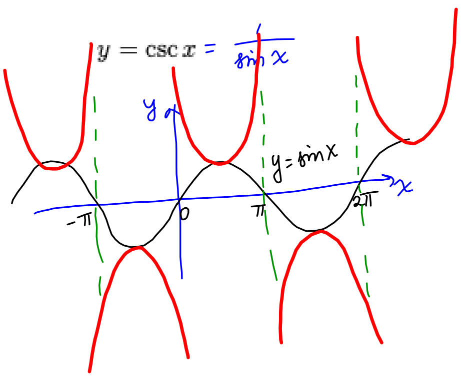
$-1 \le \cos x \le 1$

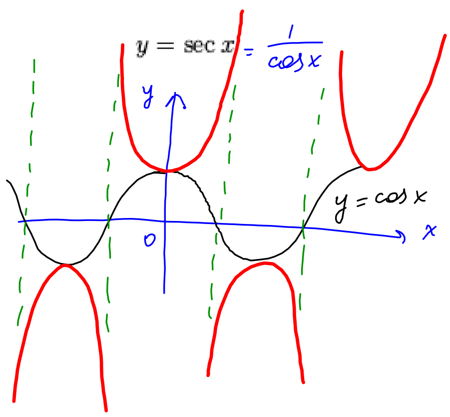
$y = \tan x = \dfrac{\sin x}{\cos x}$, $\cos x \neq 0$, $x \neq \pm \pi/2, \pm \dfrac{3\pi}{2}, \pm \dfrac{5\pi}{2}, \ldots$



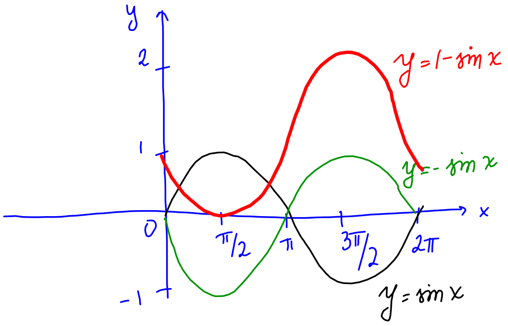
$y = \cot x = \dfrac{\cos x}{\sin x}$, $\sin x \neq 0$, $x \neq 0, \pm\pi, \pm 2\pi$




$y = \csc x = \frac{1}{\sin x}$
y
$y = \sin x$
x
$-\pi$
0
π
2π


$y = \sec x = \frac{1}{\cos x}$
y
$y = \cos x$
0
x

Example 6. Sketch the graph of $f(x) = 1 - \sin x$



1) Take the graph of $\sin x$
2) flip it about the x-axis
3) shift the flipped graph 1 unit up.

Example 7. Prove the equality:

$\sin^2 x + \cos^2 x = 1 \longrightarrow 1 - \cos^2 x = \sin^2 x$

$\tan x = \dfrac{\sin x}{\cos x}$

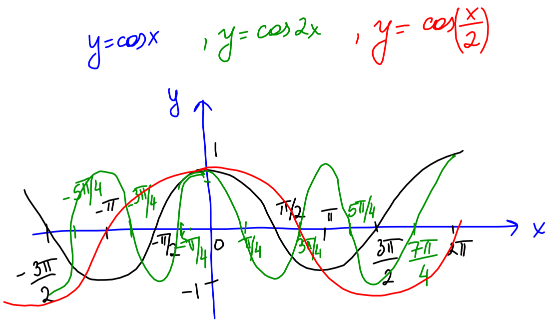
$$\tan^2 x - \sin^2 x = \tan^2 x \sin^2 x$$

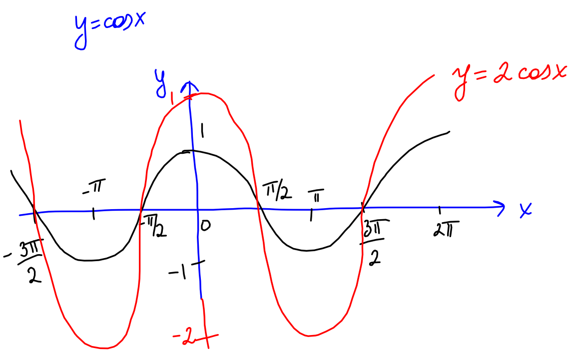
$$\tan x = \frac{\sin x}{\cos x}$$

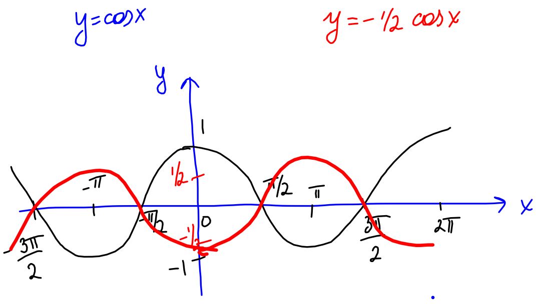
LHS: $\left(\dfrac{\sin x}{\cos x}\right)^2 - \sin^2 x = \dfrac{\sin^2 x}{\cos^2 x} - \sin^2 x = \sin^2 x \left[\dfrac{1}{\cos^2 x} - 1\right]$

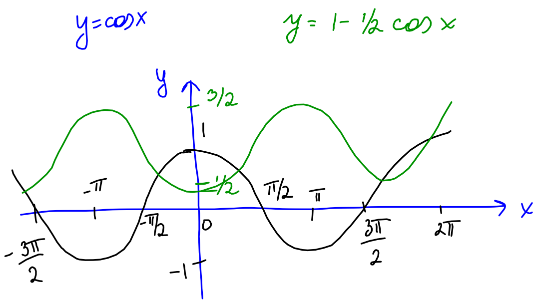
$= \sin^2 x \cdot \dfrac{1 - \cos^2 x}{\cos^2 x} = \sin^2 x \cdot \dfrac{\sin^2 x}{\cos^2 x}$

$= \sin^2 x \cdot \tan^2 x = $ RHS


y=cos x , y=cos 2x , y=cos(x/2)
y
x
1
-1
-3π/2
-π
-π/2
0
π/2
π
3π/2
2π
-5π/4
-3π/4
-π/4
π/4
3π/4
5π/4
7π/4


y=cos x
y=2 cos x
y
x
1
-1
-2
-3π/2
-π
-π/2
0
π/2
π
3π/2
2π


y=cos x
y=-1/2 cos x
y
x
1
1/2
-1/2
-1
-3π/2
-π
-π/2
0
π/2
π
3π/2
2π


y=cos x
y= 1- 1/2 cos x
y
x
3/2
1
1/2
-1
-3π/2
-π
-π/2
0
π/2
π
3π/2
2π