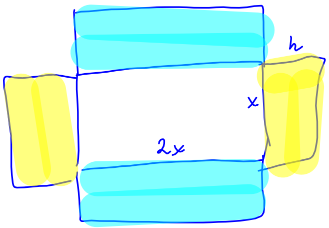
. (12 points) A rectangular storage container with a closed top is to have a volume of 10 cubic meters. The length of its base is twice the width. Material for the base costs 10 dollars per square meter, whereas material for the sides and the top costs 6 dollars per square meter. Determine the width that minimizes the cost of the container. (Your analysis must include showing that the final answer does, in fact, produce the required minimum.)



$C(x)$ is the total cost

$$C = \overbrace{(2x)\cdot x\,(10)}^{\text{base}} + \overbrace{2\,xh\,(6) + 2(2x)h(6)}^{\text{sides}} + \underbrace{x(2x)6}_{\text{top}}$$

$$= 20x^2 + 12xh + 24xh + 12x^2$$

$$= 32x^2 + 36xh \quad (\text{eliminate } h)$$

volume $\quad V = (2x)x\,h = 10$

$$2x^2h = 10 \Rightarrow h = \frac{5}{x^2}$$

$$C(x) = 32x^2 + 36x\cdot\frac{5}{x^2}$$

$$= 32x^2 + \frac{180}{x}$$

$$C'(x) = 64x - \frac{180}{x^2} = 0$$

$$64x^3 - 180 = 0, \quad x^3 = \frac{180}{64} = \frac{180}{4^3}$$

$$\boxed{x = \frac{\sqrt[3]{180}}{4}}$$

$$C''(x) = 64 - 180(-2)x^{-3}$$

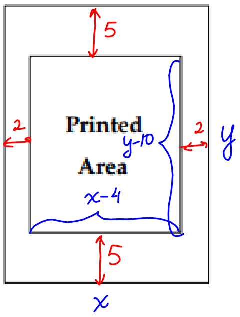
$$= 64 + \frac{360}{x^3}$$

show that $C''\left(\frac{\sqrt[3]{180}}{4}\right) > 0$

$$C''\left(\frac{\sqrt[3]{180}}{4}\right) = 64 + \frac{360}{\frac{180}{64}} = 64 + 128 > 0$$

$C(x)$ has the min @ $x = \frac{\sqrt[3]{180}}{4}$

22. (7 points) The top and bottom margins of a poster are 5cm and the side margins are each 2cm. If the area of the printed material on the posted is fixed at 250 cm^2, find the dimensions of the poster that will minimize the area of the poster. Label the variables that you use in the picture. Clearly show or explain why your answer is a minimum.



Area of the poster $A = xy$

printed area $= \underline{250 = (x-4)(y-10)}$

$$y-10 = \frac{250}{x-4}$$

$$y = 10 + \frac{250}{x-4}$$

$$A = x\left(10 + \frac{250}{x-4}\right) = 10x + \frac{250x}{x-4}$$

$$A'(x) = 10 + 250 \cdot \frac{x-4-x}{(x-4)^2} = 0$$

$$10 - \frac{1000}{(x-4)^2} = 0$$

$$1\cancel{0} = \frac{100\cancel{0}}{(x-4)^2}$$

$$(x-4)^2 = 100$$

$x-4=10$, $\quad x-4=-10$

$\underline{x=14}$, $\quad \cancel{x=-6}$

$$y = 10 + \frac{250}{14-4} = 10+25 = 35$$

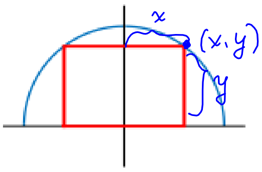
$$\boxed{14 \times 35}$$

$A(x)$ has a minimum when $x=14$

$$A''(x) = -1000(-2)(x-4)^{-3} = \frac{2000}{(x-4)^3}$$

$$A''(14) = \frac{2000}{10^3} = 2 > 0$$

17. (10 pts) A rectangle is bounded by the x-axis and the semicircle $f(x) = \sqrt{25 - x^2}$ as shown below. What length and width should the rectangle have so that its area is a maximum?



Dimensions of the rectangle are $(2x) \times y$

$y = \sqrt{25 - x^2}$

$A = (2x)y$

$A = 2x\sqrt{25 - x^2}$

$A'(x) = 2\sqrt{25-x^2} + 2x \cdot \frac{1}{2}(25-x^2)^{-1/2}(25-x^2)'$

$= 2\sqrt{25-x^2} + x(25-x^2)^{-1/2}(-2x)$

$= 2\sqrt{25-x^2} - \dfrac{2x^2}{\sqrt{25-x^2}} = 0$

2

$\sqrt{25-x^2} = \dfrac{x^2}{\sqrt{25-x^2}}$

$25 - x^2 = x^2$ or $25 = 2x^2$ or $x^2 = \dfrac{25}{2}$

$x = \dfrac{5}{\sqrt{2}}, \; y = \dfrac{5}{\sqrt{2}}$

dimensions $\boxed{\dfrac{10}{\sqrt{2}} \times \dfrac{5}{\sqrt{2}}}$

$y = \sqrt{25 - x^2} = \sqrt{25 - \left(\frac{5}{\sqrt{2}}\right)^2} = \sqrt{25 - \frac{25}{2}} = \sqrt{\frac{25}{2}}$

Critical numbers for $A(x)$ are $x = \dfrac{5}{\sqrt{2}}$, $x = 5$, $x = -5$

$A(5) = 0$ abs min

$A(-5) = 0$

$A\left(\frac{5}{\sqrt{2}}\right) = \frac{10}{\sqrt{2}}\sqrt{25 - \frac{25}{2}} = \frac{10}{\sqrt{2}} \cdot \frac{5}{\sqrt{2}} = \frac{50}{2} = \boxed{25 \text{ abs. max}}$

18. (6 pts) Newton's Law of cooling states the rate of cooling of an object is proportional to the temperature difference between the object and the temperature of the objects surroundings. A thermometer is taken from a room where the temperature is $25°C$, to the outdoors where the temperature is $-5°C$. After one minute the thermometer reads $5°C$. What will the termometer read after one more minute?

$T(t)$ temperature of the thermometer after t min

$$\frac{dT}{dt} = k(T-(-5))$$

$$\frac{dT}{dt} = k(T+5), \quad T(0)=25, \quad T(1)=5, \quad \text{find } T(2)$$

substitution: $\frac{d}{dt}u(t) = \frac{d}{dt}(T(t)+5)$ | $\frac{du}{dt} = \frac{dT}{dt} + 0$

$u(0) = 25+5 = 30$

$$\frac{du}{dt} = ku, \quad u(0)=30$$

$$u(t) = u(0)e^{kt}$$

$$u(t) = 30e^{kt}$$

$$T(t)+5 = 30e^{kt}$$

$$T(t) = 30e^{kt} - 5$$

$$T(1) = 30e^{k} - 5 = 5$$

$$30e^{k} = 10, \quad e^{k} = \frac{1}{3}, \quad k = \ln\frac{1}{3}$$

$$T(t) = 30e^{t \cdot \ln\frac{1}{3}} - 5 = \boxed{30\left(\frac{1}{3}\right)^{t} - 5 = T(t)}$$

$$T(2) = 30\left(\frac{1}{9}\right) - 5 = \frac{10}{3} - 5 = \boxed{-\frac{5}{3}}$$

2. Find $f'(e)$ if $f(x) = \ln(x + \ln x)$.

(a) $\dfrac{1}{e+1}$

(b) $\dfrac{1}{e}$

(c) $\dfrac{1}{1+e^2}$

(d) $\dfrac{1}{e^2+e}$

(e) $\dfrac{e+1}{e^2}$

$$f'(x) = \frac{1}{x+\ln x}\left(x + \ln x\right)'$$

$$= \frac{1}{x+\ln x}\left(1 + \frac{1}{x}\right)$$

$$f'(e) = \frac{1}{e+\cancel{\ln e}\,1}\left(1 + \frac{1}{e}\right)$$

$$= \frac{1}{\cancel{e+1}} \cdot \frac{\cancel{e+1}}{e} = \frac{1}{e}$$