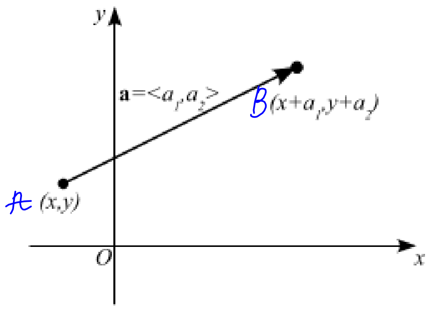
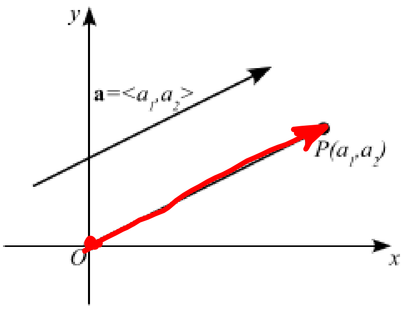
Chapter 1. **Introduction to vectors and vector functions**
Section 1.1 **Vectors**

Definition. A two-dimensional vector is an ordered pair $\vec{a} =< a_1, a_2 >$ of real numbers. The numbers a_1 and a_2 are called the **components** of $\vec{a}$.

A representation of the vector $\vec{a} =< a_1, a_2 >$ is a directed line segment $\overrightarrow{AB}$ from any point $A(x, y)$ to the point $B(x + a_1, y + a_2)$.



A particular representation of $\vec{a} =< a_1, a_2 >$ is the directed line segment $\overrightarrow{OP}$ from the origin to the point $P(a_1, a_2)$, and $\vec{a} =< a_1, a_2 >$ is called the position vector of the point $P(a_1, a_2)$.



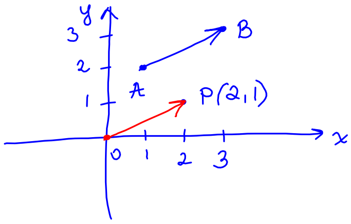
Given the points $A(x_1, y_1)$ and $B(x_2, y_2)$, then

$$\boxed{\overrightarrow{AB} =< x_2 - x_1, y_2 - y_1 >}$$

Example 1. Find a vector $\vec{a}$ with representation given by the directed line segment $\overrightarrow{AB}$. Draw $\overrightarrow{AB}$ and the equivalent representation starting at the origin.

(a) $A(1,2)$, $B(3,3)$;

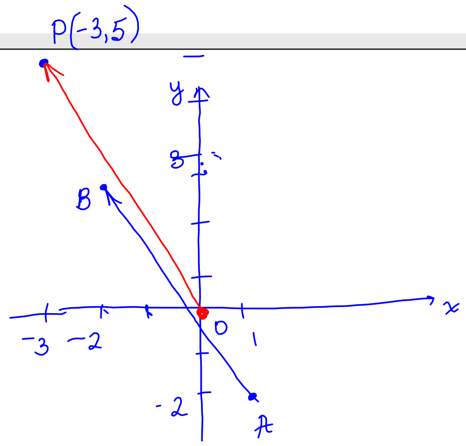
$\overrightarrow{AB} = <3-1, 3-2> = <2,1>$



(b) $A(1,-2)$, $B(-2,3)$.

$\overrightarrow{AB} = <-2-1, 3-(-2)>$

$= <-3, 5>$



The magnitude (length) $|\vec{a}|$ of $\vec{a}$ is the length of any its representation.
The length of $\vec{a} =< a_1, a_2 >$ is

$$\boxed{|\vec{a}| = \sqrt{a_1^2 + a_2^2}}$$

The length of the vector $\overrightarrow{AB}$ from $A(x_1, y_1)$ to $B(x_2, y_2)$ is

$$\boxed{|\vec{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

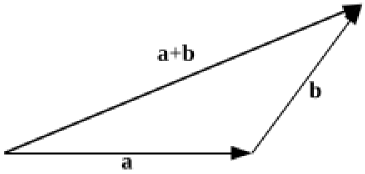
The only vector with length 0 is the **zero vector** $\boxed{\vec{0} =< 0, 0 >}$. This vector is the only vector with no specific direction.

Example 2. Find the length of the vectors from Example 1.

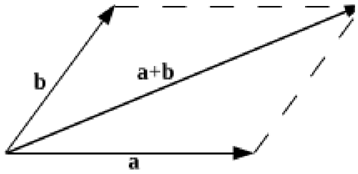
$\vec{a} = <2,1>$, $|\vec{a}| = \sqrt{2^2 + 1^2} = \sqrt{5}$

$\vec{b} = <-3,5>$, $|\vec{b}| = \sqrt{(-3)^2 + 5^2} = \sqrt{9+25} = \sqrt{34}$

Vector addition If $\vec{a} =< a_1, a_2 >$ and $\vec{b} =< b_1, b_2 >$, then the vector $\vec{a} + \vec{b}$ is defined by $\boxed{\vec{a} + \vec{b} =< a_1 + b_1, a_2 + b_2 >}$



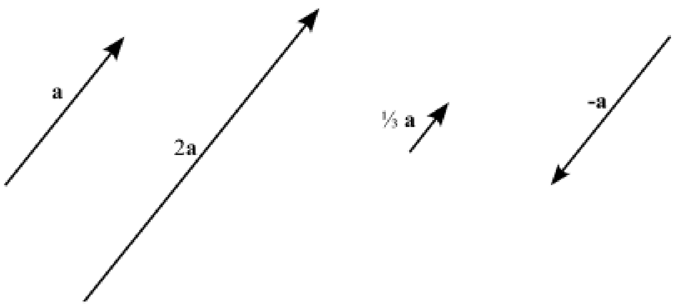
Triangle Law



Parallelogram Law

Multiplication of a vector by a scalar If c is a scalar and $\vec{a} =< a_1, a_2 >$, then the vector is defined by

$$\boxed{c\vec{a} =< ca_1, ca_2 >}$$



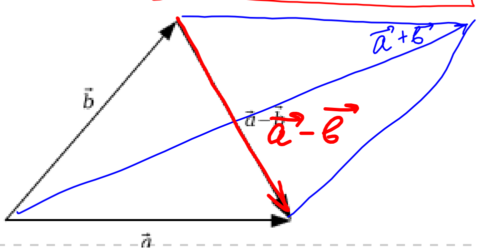
$$|c\vec{a}| = |c||\vec{a}|$$

Two vectors $\vec{a}$ and $\vec{b}$ are called parallel if $\vec{b} = c\vec{a}$ for some scalar c.

By the difference of two vectors, we mean

$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$$

so, if $\vec{a} =< a_1, a_2 >$ and $\vec{b} =< b_1, b_2 >$, then $\boxed{\vec{a} - \vec{b} =< a_1 - b_1, a_2 - b_2 >}$.



Example 3. If $\vec{a} =< -1, 2 >$ and $\vec{b} =< -2, -1 >$, find

(a) $\vec{a} + \vec{b} = < -1, 2 > + < -2, -1 > = < -3, 1 >$

(b) $1/2\vec{b} = \frac{1}{2} < -2, -1 > = < -\frac{2}{2}, -\frac{1}{2} > = < -1, -\frac{1}{2} >$

(c) $\vec{a} - \vec{b} = < -1, 2 > - < -2, -1 > = < -1-(-2), 2-(-1) > = < 1, 3 >$

(d) $|2\vec{a} - 5\vec{b}|$

$$2\vec{a} - 5\vec{b} = 2< -1, 2 > - 5< -2, -1 >$$
$$= < -2, 4 > - < -10, -5 > = < 8, 9 >$$
$$|2\vec{a} - 5\vec{b}| = \sqrt{8^2 + 9^2} = \sqrt{145}$$

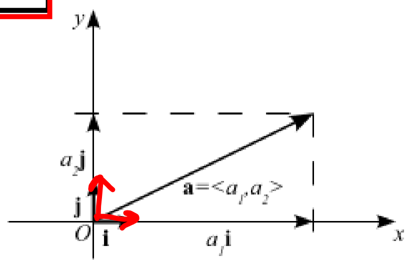
Properties of vectors. If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors and k and m are scalars, then

1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
2. $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$
3. $\vec{a} + \vec{0} = \vec{a}$
4. $\vec{a} + (-\vec{a}) = \vec{0}$
5. $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$
6. $(k + m)\vec{a} = k\vec{a} + m\vec{a}$
7. $(km)\vec{a} = k(m\vec{a})$ $= m(k\vec{a})$
8. $1\vec{a} = \vec{a}$

Let $\vec{\imath} =< 1, 0 >$ and $\vec{\jmath} =< 0, 1 >$.

$$|\vec{\imath}| = |\vec{\jmath}| = 1$$

$$\boxed{\vec{a} =< a_1, a_2 >= a_1\vec{\imath} + a_2\vec{\jmath}}$$



Example 4. Express $\vec{a} =< 2, 4 >$, $\vec{b} =< -1, 3 >$, and $2\vec{a} + \vec{b}$ in terms of $\vec{\imath}$ and $\vec{\jmath}$.

$\vec{a} = <2,4> = 2\vec{\imath} + 4\vec{\jmath}$

$\vec{b} = <-1,3> = -1\vec{\imath} + 3\vec{\jmath} = -\vec{\imath} + 3\vec{\jmath}$

$2\vec{a} + \vec{b} = 2(\underbrace{2\vec{\imath} + 4\vec{\jmath}}_{\vec{a}}) + (\underbrace{-\vec{\imath} + 3\vec{\jmath}}_{\vec{b}})$

$= 4\vec{\imath} + 8\vec{\jmath} - \vec{\imath} + 3\vec{\jmath} = 3\vec{\imath} + 11\vec{\jmath}$

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A unit vector is a vector whose length is 1.

A vector

$$\boxed{\vec{u} = \frac{1}{|\vec{a}|}\vec{a} = \left\langle \frac{a_1}{|\vec{a}|}, \frac{a_2}{|\vec{a}|} \right\rangle}$$

is a unit vector that has the same direction as $\vec{a} =< a_1, a_2 >$.

Example 5. Given vectors $\vec{a} = \vec{\imath} - 2\vec{\jmath}$, $\vec{b} =< -2, 3 >$. Find a unit vector $\vec{u}$ that has the same direction as $2\vec{b} + \vec{a}$.

$\vec{a} = < 1, -2 >$

$2\vec{b} + \vec{a} = 2<-2,3> + <1,-2> = <-4+1, 6-2> = <-3,4>$

$\vec{u} = \dfrac{2\vec{b}+\vec{a}}{|2\vec{b}+\vec{a}|} = \dfrac{<-3,4>}{\sqrt{(-3)^2+4^2}} = \dfrac{<-3,4>}{\sqrt{9+16}} = \dfrac{1}{5}<-3,4> = \boxed{<-\frac{3}{5}, \frac{4}{5}>}$

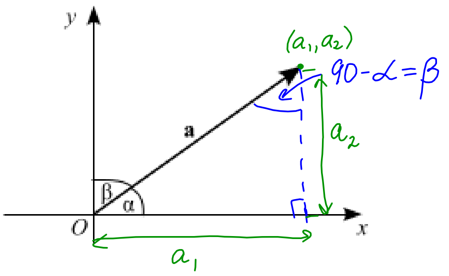
Find a vector of length 4 in the direction of $\vec{b}$.

unit vector in the direction of b: $\vec{u} = \dfrac{\vec{b}}{|\vec{b}|} = \dfrac{<-2,3>}{\sqrt{(-2)^2+3^2}}$

$= \dfrac{<-2,3>}{\sqrt{13}} = <-\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}>$ – unit

$4\dfrac{\vec{b}}{|\vec{b}|} = 4<-\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}> = \boxed{<-\frac{8}{\sqrt{13}}, \frac{12}{\sqrt{13}}>}$

Direction angles and direction cosines. The direction angles of a nonzero vector $\vec{a}$ are the angles α and β in the interval $[0, \pi]$ that $\vec{a}$ makes with the positive $x-$ and $y-$ axes. The cosines of these direction angles, $\cos\alpha$ and $\cos\beta$ are called the **direction cosines** of the vector $\vec{a}$.



$\cos\alpha = \dfrac{a_1}{|\vec{a}|}$

$\cos\beta = \dfrac{a_2}{|\vec{a}|} = \sin\alpha$

$$\cos\alpha = \frac{a_1}{|\vec{a}|}, \quad \cos\beta = \frac{a_2}{|\vec{a}|}, \quad \cos^2\alpha + \cos^2\beta = 1$$

$\vec{a} = < a_1, a_2 >$

$= |\vec{a}| < \dfrac{a_1}{|\vec{a}|}, \dfrac{a_2}{|\vec{a}|} >$

$= \boxed{|\vec{a}| < \cos\alpha, \sin\alpha > = \vec{a}}$

We can write

$$\vec{a} =< a_1, a_2 >= |\vec{a}| < \cos\alpha, \overset{\sin\alpha}{\cancel{\cos\beta}} >$$

Therefore

$$\frac{1}{|\vec{a}|}\vec{a} =< \cos\alpha, \overset{\sin\alpha}{\cancel{\cos\beta}} >$$

[redacted]

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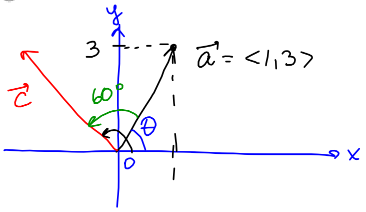
Example 6. Let $\vec{c}$ be the vector obtained by rotating $\vec{a} =< 1, 3 >$ by an angle of 60 degrees in the counterclockwise direction. Compute the vector $\vec{c}$.



$|\vec{a}| = |\vec{c}| = \sqrt{1+3^2} = \sqrt{10}$

$\vec{a} = |\vec{a}| < \dfrac{a_1}{|\vec{a}|}, \dfrac{a_2}{|\vec{a}|} > = \sqrt{10} < \underbrace{\dfrac{1}{\sqrt{10}}}_{\cos\theta}, \underbrace{\dfrac{3}{\sqrt{10}}}_{\sin\theta} >$

$\vec{c} = |\vec{c}| < \cos(\theta+60°), \sin(\theta+60°) >$

$\cos(\theta+60°) = \cos\theta\cos60° - \sin\theta\sin60° = \overbrace{\dfrac{1}{\sqrt{10}}}^{\cos\theta} \cdot \dfrac{1}{2} - \overbrace{\dfrac{3}{\sqrt{10}}}^{\sin\theta} \dfrac{\sqrt{3}}{2} = \dfrac{1-3\sqrt{3}}{2\sqrt{10}}$

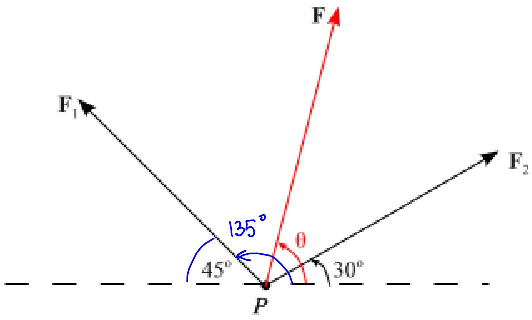
$\cos(x+y) = \cos x\cos y - \sin x \sin y$

$\sin(\theta+60°) = \sin\theta\cos60° + \cos\theta\sin60° = \dfrac{3}{\sqrt{10}} \cdot \dfrac{1}{2} + \dfrac{1}{\sqrt{10}} \cdot \dfrac{\sqrt{3}}{2} = \dfrac{3+\sqrt{3}}{2\sqrt{10}}$

$\sin(x+y) = \sin x\cos y + \cos x \sin y$

$\vec{c} = \cancel{\sqrt{10}} < \dfrac{1-3\sqrt{3}}{2\cancel{\sqrt{10}}}, \dfrac{3+\sqrt{3}}{2\cancel{\sqrt{10}}} > = \boxed{< \dfrac{1-3\sqrt{3}}{2}, \dfrac{3+\sqrt{3}}{2} >}$

Example 7. Two forces $\vec{F}_1$ and $\vec{F}_2$ with magnitudes 10 lb and 12 lb act on an object at a point P as shown in the figure. Find the resultant force $\vec{F}$ acting at P as well as its magnitude and its direction.



$\vec{F} = \vec{F}_1 + \vec{F}_2$

$|\vec{F}_1| = 10, \quad |\vec{F}_2| = 12$

$\vec{F}_1 = |\vec{F}_1| < \overbrace{\cos 135}^{-\cos 45°}, \overbrace{\sin 135}^{\sin 45} >$

$= 10 < -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} >$

$\vec{F}_1 = < -5\sqrt{2}, 5\sqrt{2} >$

$\vec{F}_2 = |\vec{F}_2| < \cos 30°, \sin 30° >$

$= 12 < \frac{\sqrt{3}}{2}, \frac{1}{2} >$

$\vec{F}_2 = < 6\sqrt{3}, 6 >$

$\vec{F} = \vec{F}_1 + \vec{F}_2 = < -5\sqrt{2}, 5\sqrt{2} > + < 6\sqrt{3}, 6 >$

$= < -5\sqrt{2} + 6\sqrt{3}, 5\sqrt{2} + 6 > \approx < 3.3, 13.07 >$

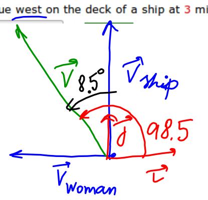
$|\vec{F}| = \sqrt{(3.3)^2 + (13.07)^2} \approx 13.5$

$\tan\theta = \frac{13.07}{3.3} \approx 3.96$

$\theta = \arctan(3.96) \approx \ldots$

$\vec{v}$ is the velocity, $|\vec{v}|$ is the speed

A woman walks due west on the deck of a ship at 3 mi/h. The ship is moving north at a speed of 20 mi/h. Find the speed and direction of the woman relative to the surface of the water. (Round your answers to one decimal place.)



$\vec{v}_{woman} = 3\langle -1, 0\rangle = \langle -3, 0\rangle$

$|\vec{v}_{woman}| = 3$

$|\vec{v}_{ship}| = 20$

$\vec{v}_{ship} = 20\langle 0, 1\rangle = \langle 0, 20\rangle$

$\vec{v} = \vec{v}_{woman} + \vec{v}_{ship} = \langle -3, 0\rangle + \langle 0, 20\rangle = \langle -3, 20\rangle$

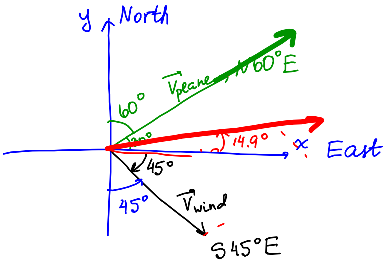
speed $= |\vec{v}| = \sqrt{3^2 + 20^2} = \sqrt{409} \approx \boxed{20.2}$ magnitude

$\tan\theta = \frac{20}{-3} \approx -6.67$

$\theta = \arctan(-6.67) \approx 98.5$

direction $N(98.5 - 90)W = \boxed{N(8.5°)W}$ direction

Suppose that a wind is blowing in the direction S45°E at a speed of 30 km/h. A pilot is steering a plane in the direction N60°E at an airspeed (speed in still air) of 100 km/h. The *true course*, or *track*, of the plane is the direction of the resultant of the velocity vectors of the plane and the wind. The *ground speed* of the plane is the magnitude of the resultant. Find the true course and the ground speed of the plane. (Round your answers to one decimal place.)



$|\vec{v}_{wind}| = 30$

$|\vec{V}_{plane}| = 100$

$\vec{V}_{wind} = 30 < \cos 45°, -\sin 45° >$

$= 30 < \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} > = < 15\sqrt{2}, -15\sqrt{2} >$

$\vec{V}_{plane} = 100 < \cos 30°, \sin 30° >$

$= 100 < \frac{\sqrt{3}}{2}, \frac{1}{2} > = < 50\sqrt{3}, 50 >$

$\vec{V} = \vec{V}_{wind} + \vec{V}_{plane} = < 15\sqrt{2}, -15\sqrt{2} > + < 50\sqrt{3}, 50 >$

$= < 15\sqrt{2} + 50\sqrt{3}, 50 - 15\sqrt{2} >$

ground speed $= |\vec{V}| = \sqrt{(15\sqrt{2} + 50\sqrt{3})^2 + (50 - 15\sqrt{2})^2} = \ldots$

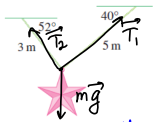
$\tan\theta = \frac{50 - 15\sqrt{2}}{15\sqrt{2} + 50\sqrt{3}}$, $\theta = 14.9$

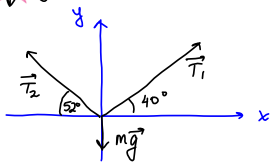
direction $N(90 - 14.9)E$

$\boxed{N(75.1°)E}$

Ropes 3 m and 5 m in length are fastened to a holiday decoration that is suspended over a town square. The decoration has a mass of 4 kg. The ropes, fastened at different heights, make angles of 52° and 40° with the horizontal. Find the magnitude of the tension in each wire. (Use $g = 9.8$ m/s^2 for the acceleration due to gravity. Round your answers to two decimal places.)

Magnitude of tension in 3 m rope		30.05	N
Magnitude of tension in 5 m rope		24.15	N





Find $|\vec{T_1}|, |\vec{T_2}|$ - ?

$$\vec{T_1} + \vec{T_2} = m\vec{g}$$

$$\vec{T_1} = |\vec{T_1}| \langle \cos 40°, \sin 40° \rangle$$

$$\vec{T_2} = |\vec{T_2}| \langle -\cos 52°, \sin 52° \rangle$$

$$m\vec{g} = \langle 0, +4(9.8) \rangle$$

$$|\vec{T_1}| \langle \cos 40°, \sin 40° \rangle + |\vec{T_2}| \langle -\cos 52°, \sin 52° \rangle = \langle 0, +4(9.8) \rangle$$

Equate the corresponding components:

$$\begin{cases} |\vec{T_1}| \cos 40° - |\vec{T_2}| \cos 52° = 0 \\ |\vec{T_1}| \sin 40° + |\vec{T_2}| \sin 52° = (9.8)4 \end{cases}$$

solve for $|\vec{T_1}|$ and $|\vec{T_2}|$

1st eqn. $|\vec{T_1}| = \dfrac{|\vec{T_2}| \cos 52°}{\cos 40°}$ plug into the 2nd eqn.

$$\frac{|\vec{T_2}| \cos 52°}{\cos 40°} \sin 40° + |\vec{T_2}| \sin 52° = 4(9.8)$$

$$|\vec{T_2}| (\cos 52° \tan 40° + \sin 52°) = 4(9.8)$$

$$|\vec{T_2}| = \frac{4(9.8)}{\cos 52° \tan 40° + \sin 52°}$$

, plug $|\vec{T_2}|$ into the equation for $|\vec{T_1}|$