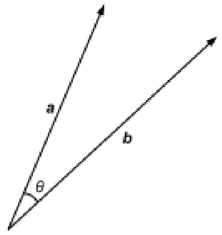
Section 1.2 **The dot product**

Definition. The dot or scalar product of two nonzero vectors $\vec{a}$ and $\vec{b}$ is the number

$$\boxed{\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta}$$

where θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \leq \theta \leq \pi$. If either $\vec{a}$ or $\vec{b}$ is $\vec{0}$, we define $\vec{a} \cdot \vec{b} = 0$.

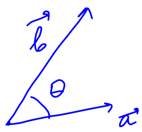


$\vec{a} \cdot \vec{b} \geq 0$, if $0 \leq \theta < \pi/2$

$\vec{a} \cdot \vec{b} \leq 0$, if $\frac{\pi}{2} < \theta \leq \pi$

Example 1. If the vectors $\vec{a}$ and $\vec{b}$ have lengths 2 and 6, and the angle between them is $\pi/4$, find $\vec{a} \cdot \vec{b}$.

$|\vec{a}| = 2$, $\vec{b} = 6$, $\theta = \pi/4$



$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

$$= 2(6) \cos\frac{\pi}{4} = 2(6)\frac{\sqrt{2}}{2}$$

$$= \boxed{6\sqrt{2}}$$

If $\vec{a} =< a_1, a_2 >$ and $\vec{b} =< b_1, b_2 >$, then

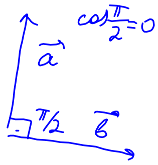
$$\boxed{\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2}$$

Example 2. Find $\vec{a} \cdot \vec{b}$ if $\vec{a} =< 2, 3 >$ and $\vec{b} = \vec{\imath} - 3\vec{\jmath}$

$=< 1, -3 >$

$$\vec{a} \cdot \vec{b} = 2(1) + 3(-3) = \boxed{-7}$$

Two nonzero vectors $\vec{a}$ and $\vec{b}$ are called perpendicular or orthogonal if the angle between them is $\pi/2$.



Two vectors $\vec{a}$ and $\vec{b}$ are orthogonal if and only if $\vec{a} \cdot \vec{b} = 0$.

Example 3. Find the angle between the vectors $\vec{a} = 6\vec{\imath} - 2\vec{\jmath}$ and $\vec{b} =< 1, 1 >$.

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta \longrightarrow \boxed{\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}}$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{6(1) - 2(1)}{\sqrt{6^2 + (-2)^2}\sqrt{1+1}} = \frac{4}{\sqrt{2} \cdot \sqrt{40}} = \boxed{\frac{1}{\sqrt{5}}}$$

$$\theta = \arccos\left(\frac{1}{\sqrt{5}}\right) \approx \boxed{63.4^\circ}$$

Example 4. Determine whether the given vectors are orthogonal, parallel, or neither.

$\vec{a} = \langle a_1, a_2 \rangle$, $\vec{b} = \langle b_1, b_2 \rangle$ are parallel if and only if $\boxed{\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2}}$

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perpendicular $\boxed{\vec{a} \cdot \vec{b} = 0}$

(a.) $\vec{a} = \langle 1, -2 \rangle$, $\vec{b} = \underbrace{-2\vec{\imath} + 4\vec{\jmath}}_{}$ $= \langle -2, 4 \rangle$

$\vec{a} \cdot \vec{b} = 1(-2) - 2(4) = -10 \neq 0$ not orthogonal

$\dfrac{1}{-2} \overset{?}{=} \dfrac{-2}{4}$ $\boxed{-\dfrac{1}{2} = -\dfrac{1}{2}}$ $\boxed{\text{parallel}}$

(b.) $\vec{a} = \langle 3, 1 \rangle$, $\vec{b} = \langle -3, 9 \rangle$

$\vec{a} \cdot \vec{b} = 3(-3) + 1(9) = 0$

$\boxed{\text{orthogonal}}$

(c.) $\vec{a} = -\vec{\imath} + 4\vec{\jmath}$ $\langle -1, 4 \rangle$, $\vec{b} = 3\vec{\imath} - 2\vec{\jmath}$ $\langle 3, -2 \rangle$

$\vec{a} \cdot \vec{b} = -1(3) + 4(-2) = -11 \neq 0$

$-\dfrac{1}{3} \neq \dfrac{4}{-2}$

$\boxed{\text{neither parallel nor perpendicular}}$

Properties of the dot product If $\vec{a}$, $\vec{b}$, an $\vec{c}$ are vectors and k is a scalar, then

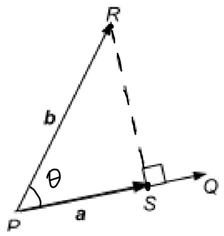
1. $\vec{a} \cdot \vec{a} = |\vec{a}|^2$
2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
4. $(k\vec{a}) \cdot \vec{b} = k(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (k\vec{b})$
5. $\vec{0} \cdot \vec{a} = 0$

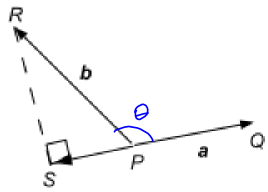
$0 < \theta < \frac{\pi}{2}$

ΔPRS

$|\vec{PS}| = |\vec{b}| \cos\theta$

$= \cancel{|\vec{b}|} \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cancel{|\vec{b}|}}$





$\frac{\pi}{2} < \theta < \pi$

$\vec{a}\cdot\vec{b} < 0$

$\text{comp}_{\vec{a}}\vec{b} < 0$

$\boxed{\text{comp}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}$

$\boxed{\text{proj}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a}}$

$\vec{PS}$ $=\text{proj}_{\vec{a}}\vec{b}$ is called the **vector projection of $\vec{b}$ onto $\vec{a}$.**

$|\vec{PS}|$ $=\text{comp}_{\vec{a}}\vec{b}$ is called the **scalar projection of $\vec{b}$ onto $\vec{a}$** or the **component of $\vec{b}$ along** $\vec{a}$.

$$\boxed{\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}$$

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$$\vec{PS} = |\vec{PS}| \overbrace{\frac{\vec{a}}{|\vec{a}|}}^{\text{unit vector in the direction of } \vec{a}} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\,\frac{\vec{a}}{|\vec{a}|}$$

$$\boxed{\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a}}$$

Example 5. Find the scalar and the vector projections of $\vec{b} = <4, 2>$ onto $\vec{a} = \overset{<1,1>}{\vec{\imath} + \vec{\jmath}}$.

$$\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{4(1)+2(1)}{\sqrt{1+1}} = \boxed{\frac{6}{\sqrt{2}}}$$

$$\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a} = \frac{6}{2}<1,1> = \boxed{<3,3>}$$

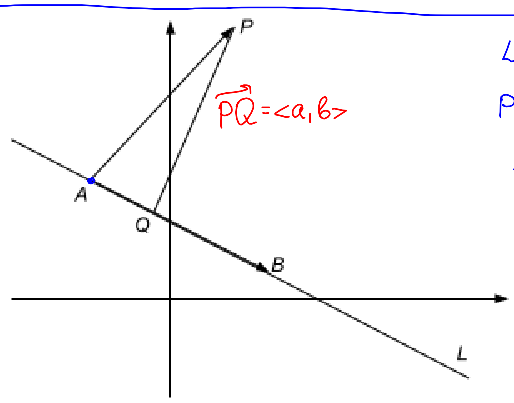
if a force $\vec{F}$ moves an object along a straight line from A to B, then the work done by the force

$$\boxed{W = \vec{F}\cdot\vec{AB}}$$

Definition. Given the nonzero vector $\vec{a} =< a_1, a_2 >$, the orthogonal complement of $\vec{a}$ is the vector $\vec{a}^{\perp} =< -a_2, a_1 >$. or $\vec{a}^{\perp} = \langle a_2, -a_1 \rangle$

$\vec{a} \cdot \vec{a}^{\perp} = 0$

Vectors $\vec{a}$ and $\vec{a}^{\perp}$ are orthogonal and $|\vec{a}| = |\vec{a}^{\perp}|$



$\vec{PQ} = \langle a, b \rangle$

$L: ax + by + c = 0$

$\langle a, b \rangle$ is perpendicular to the line

$P(x_1, y_1)$

A is an arbitrary point on the line L.

The distance from the point P to the line L

$$|PQ| = \text{comp}_{\overrightarrow{AB}^{\perp}} \overrightarrow{AP}$$

$$\boxed{d = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|}$$

Example 6. Find the distance from the point $(\overbrace{0}^{x_1},\overbrace{4}^{y_1})$ to the line $2x + 5y = -3$.

$$\overbrace{2}^{a}x + \overbrace{5}^{b}y \overbrace{+3}^{c} = 0$$

$$d = \left| \frac{2(0) + 5(4) + 3}{\sqrt{2^2 + 5^2}} \right| = \boxed{\frac{23}{\sqrt{29}}}$$