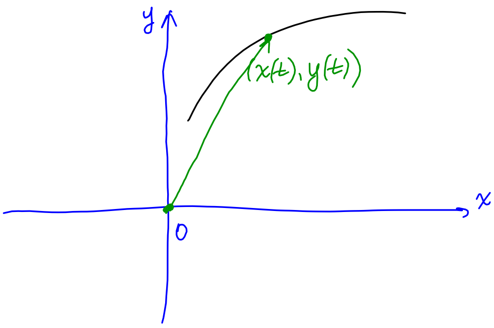
Section 1.3 **Vector functions.**

Definition. The curve of a type $x = x(t)$, $y = y(t)$ is called a **parametric curve** and the variable t is called a **parameter**.

Definition. Vector $\boxed{\vec{r}(t) =< x(t), y(t) >} = x(t)\vec{\imath} + y(t)\vec{\jmath}$ is called the **position vector** for the point with coordinates $(x(t), y(t))$.

A function such as $\vec{r}(t)$, whose range is a set of vectors, is called **a vector function** of t.



parameter t

$$\left.\begin{cases} x = x(t) \\ y = y(t) \end{cases}\right\}$$ parametric equations of the curve.

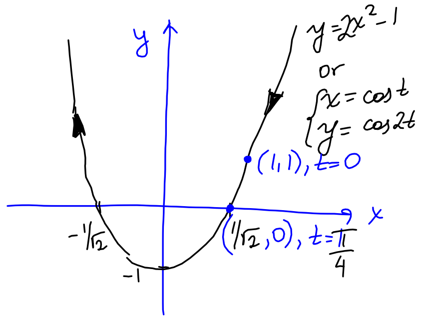
Position vector of $(x(t), y(t))$

$\vec{r}(t) = < x(t), y(t) >$

Example 1.

(a.) Sketch the curve represented by the parametric equations $\cancel{x(t) = \frac{1-t}{1+t},\ y = t^2}$.

$x(t) = \cos t, \quad y(t) = \cos 2t, \quad 0 \le t \le 2\pi$



$t=0$: $x(0) = \cos 0 = 1$

$y(0) = \cos 0 = 1$

$t = \frac{\pi}{4}$: $x\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$y\left(\frac{\pi}{4}\right) = \cos 2\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{2} = 0$

(b.) Eliminate the parameter to find the Cartesian equation of the curve.

$x = \cos t, \quad y = \cos 2t = 2\cos^2 t - 1$

$y = 2x^2 - 1$ parabola.

$$\begin{cases} x(t) = 2\cos\theta \\ y(t) = 3\sin\theta. \end{cases} \qquad 0 \le \theta \le 2\pi.$$

$\cos^2 x + \sin^2 x = 1$

Sketch the curve and find its Cartesian equation.

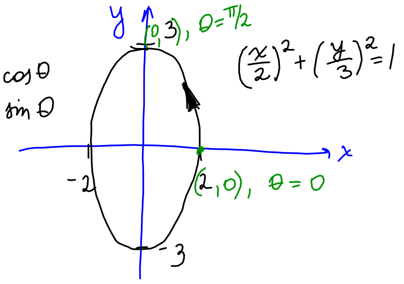
$$\frac{x}{2} = \frac{2\cos\theta}{2} \Rightarrow \cos\theta = \frac{x}{2}$$

$$\frac{y}{3} = \frac{3\sin\theta}{3} \Rightarrow \sin\theta = \frac{y}{3}$$

$\cos^2\theta + \sin^2\theta = 1$

$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$ – ellipse.

$x = 2\cos\theta$
$y = 3\sin\theta$



$\theta = 0$: $x = 2\cos 0 = 2$
$y = 3\sin 0 = 0$

$\theta = \frac{\pi}{2}$: $x = 2\cos \pi/2 = 0$
$y = 3\sin \pi/2 = 3$

Example 2. An object is moving in the xy-plane and its position after t seconds is $\vec{r}(t) =< t-3, t^2-2t >$.

(a.) Find the position of the object at time $t = 5$.

$$\vec{r}(5) = < 5-3,\ 5^2 - 2(5) > = \boxed{< 2, 15 >}$$

(b.) At what time is the object at the point (1,8).

Find t such that $\vec{r}(t) = < 1, 8 >$

$$< t-3,\ t^2-2t > = < 1, 8 >$$

$$\begin{cases} t-3 = 1 \Rightarrow \boxed{t=4} \\ t^2 - 2t = 8 \end{cases}$$

(c.) Does the object pass through the point (3,20).

$$< t-3,\ t^2-2t > = < 3, 20 >$$

$$\begin{cases} t-3 = 3 \longrightarrow t = 6 \\ t^2 - 2t = 20 \end{cases}$$

plug $t=6$ into the 2nd equation:

$$6^2 - 2(6) = 36 - 12 = 24 \neq 20$$

$\boxed{NO}$

(d.) Find an equation in x and y whose graph is the path of the object.

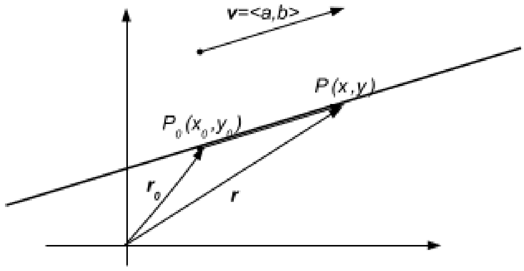
$x = t-3$, $y = t^2 - 2t$

$t = x+3$

$$\boxed{y = (x+3)^2 - 2(x+3)}$$

Problem: Write down an equation of a line through $P_0(x_0, y_0)$ parallel to the vector $\vec{v} = <a, b>$

A line L is determined by a point P_0 on L and a direction. Let $\vec{v}$ be a vector parallel to line L. Let P be be an arbitrary point on L and let $\vec{r_0}$ and $\vec{r}$ be the position vectors of P and P_0.



Then the *vector equation* of line L is

$$\vec{r}(t) = \vec{r_0} + t\vec{v}$$

$$\boxed{\vec{r}(t) = <x_0, y_0> + t<a, b>}$$

$$\vec{r}(t) = <x(t), y(t)> = <x_0 + ta, y_0 + tb>$$

If $\vec{r} =< x(t), y(t) >$, $\vec{v} =< a, b >$ and $P(x_0, y_0)$ then *parametric equations* of the line L are

$$\boxed{x(t) = x_0 + at,} \quad \boxed{y(t) = y_0 + bt}$$

Example 3. Find a vector, parametric, and Cartesian equations for the line containing the point (2,-1) and parallel to $2\vec{\imath} + 3\vec{\jmath}$.

$<2, 3>$

vector equation: $\vec{r}(t) = <2, -1> + t<2, 3>$

parametric equations: $<x(t), y(t)> = <2, -1> + t<2, 3>$

$$\begin{cases} x(t) = 2 + 2t \longrightarrow t = \dfrac{x-2}{2} \\ y(t) = -1 + 3t \end{cases}$$

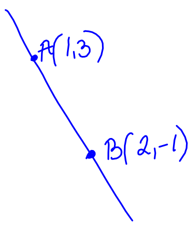
Cartesian equation: eliminate t.

$$2(y) = \left(-1 + 3\,\frac{x-2}{2}\right)2 \Rightarrow 2y = -2 + 3(x-2)$$

$$-2 + 3x - 6 - 2y = 0$$

$$\boxed{3x - 2y - 8 = 0}$$

Example 4. Find a vector and parametric equations for the line passing through the points $A(1, 3)$ and $B(2, -1)$.



$\vec{AB}$ is parallel to the line.

$\vec{AB} = <2-1, -1-3>$

$= <1, -4> = \vec{v}$

$$\boxed{\begin{array}{l} \vec{r}(t) = <1, 3> + t<1, -4> \\ \begin{cases} x(t) = 1 + t \\ y(t) = 3 - 4t \end{cases} \end{array}} \text{ or } \boxed{\begin{array}{l} \vec{r}(t) = <2, -1> + t<1, -4> \\ \begin{cases} x(t) = 2 + t \\ y(t) = -1 - 4t \end{cases} \end{array}}$$