Chapter 2. **Limits and rates of change**
Section 2.5 **Continuity**

Definition. A function f is **continuous at a number** a if

$$\lim_{x\to a} f(x) = f(a)$$

If f is not continuous at a, then f has **discontinuity** at a:

- if $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$, then f has a **jump discontinuity** at a,
- if either $\lim_{x\to a^+} f(x) = \infty$ or $\lim_{x\to a^-} f(x) = \infty$, then f has an **infinity discontinuity** at a and we say line $x = a$ is a **vertical asymptote** of the curve $y = f(x)$
- if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) \neq f(a)$, then f has a **removable discontinuity** at a

Example 1. Show that function $f(x) = x^2 + 2x + 3$ is continuous at $a = 2$.

Example 2. Explain why the function

$$f(x) = \begin{cases} \dfrac{1}{(x-1)^2}, & \text{if } x \neq 1 \\ 0, & \text{if } x = 1 \end{cases}$$

is discontinuous at $a = 1$. Sketch the graph of the function.

Example 3. Find the points at which f is discontinuous.

$$f(x) = \begin{cases} \dfrac{1}{x}, & \text{if } x < -1 \\ x, & \text{if } -1 \leq x < 1 \\ \dfrac{1}{x^2}, & \text{if } x \geq 1 \end{cases}$$

Definition. A function f is **continuous from the right at a number** a if

$$\lim_{x \to a^+} f(x) = f(a)$$

f is **continuous from the left at a number** a if

$$\lim_{x \to a^-} f(x) = f(a)$$

Definition A function f is **continuous on an interval** if it is continuous at every number in the interval. (At an endpoint of the interval we understand **continuous** to mean **continuous from the right** or **continuous from the left**.)

Example 4. Show that the function $f(x) = x\sqrt{16 - x^2}$ is continuous on its domain. State the domain.

Example 5. For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx + 1, & \text{if } x \leq 3 \\ cx^2 - 1, & \text{if } x > 3 \end{cases}$$

Theorem. If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$ 2. $f + g$ 3. cf 4. fg 5. $\frac{f}{g}$ if $g(a) \neq 0$

Theorem.

(a.) Any polynomial is continuous on $(-\infty, \infty)$

(b.) Any rational function is continuous on its domain

Theorem. If n is a positive even integer, then $f(x) = \sqrt[n]{x}$ is continuous on $[0, \infty)$. If n is a positive odd integer, then f is continuous on $(-\infty, \infty)$.

Example 6. On what interval is the function $h(x) = \sqrt{x} + \dfrac{1}{x-2} - \dfrac{1+2x}{x^2+4}$ continuous?

Theorem. If f is continuous at b and $\lim\limits_{x \to a} g(x) = b$, then

$$\lim_{x \to a} f(g(x)) = f(b) = f\left(\lim_{x \to a} g(x)\right)$$

Theorem. If g is continuous at a and f is continuous at $g(a)$, then $(f \circ g)(x) = f(g(x))$ is continuous at a.

The intermediate value theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number strictly between $f(a)$ and $f(b)$. Then there exist a number c in (a, b) such that $f(c) = N$.

Example 7. Use the intermediate value theorem to show that there is a root of the equation $x^3 + 2x = x^2 + 1$ in the interval (0,1).