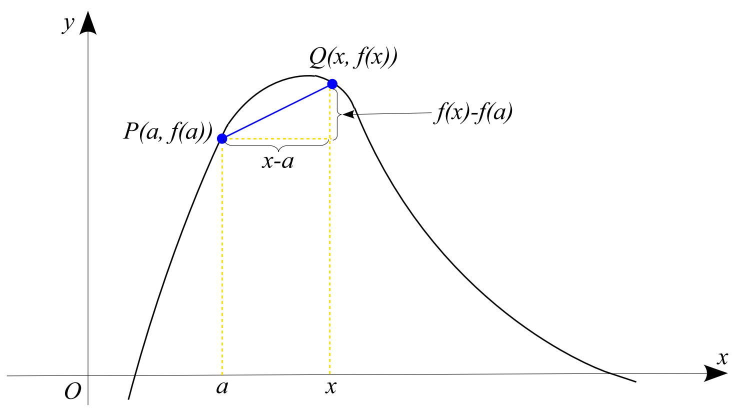
Section 2.7 **Tangents, velocities, and other rates of change**

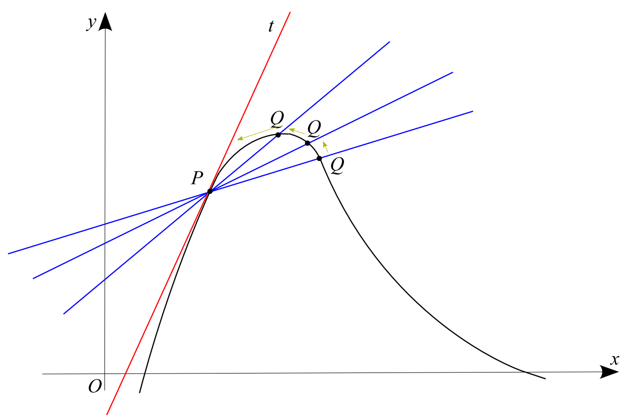
The tangent line.

If a curve C has equation $y = f(x)$ and we want to find the tangent to C at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line PQ:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$



Then we let Q approach P along the curve C by letting x approach a.



If m_{PQ} approaches a number m, then we define the **tangent** t to be the line through P with slope m.

Definition. The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

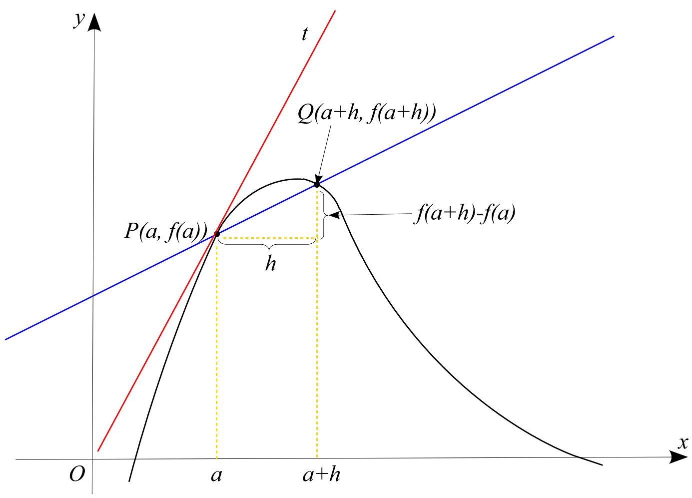
$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists. Then the equation of the tangent line is

$$y = m(x - a) + f(a)$$

Let $h = x - a$, then $x = a + h$, so the slope of the secant line PQ is

$$m_{PQ} = \frac{f(a+h) - f(a)}{h}$$



Then the slope of the tangent line becomes

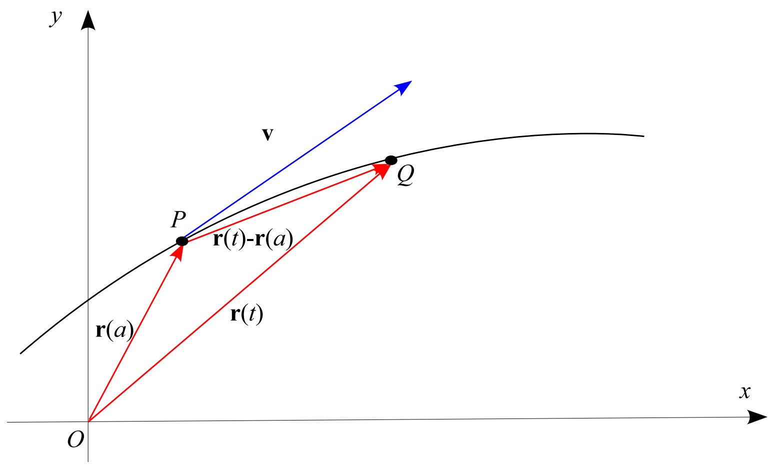
$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Example 1. Find the equation of the tangent line to the curve $y = \sqrt{2x-3}$ at the point (2,1).

Tangent vectors

Let $\vec{r}(t) =< x(t), y(t) >$ be a vector function.

Problem. Find a tangent vector to a curve traced by $\vec{r}(t)$ at the point P corresponding to the vector $\vec{r}(a) =< x(a), y(a) >$.



The tangent vector to a curve traced by $\vec{r}(t)$ at the point P corresponding to the vector $\vec{r}(a) =< x(a), y(a) >$ is given by

$$\vec{v} = \lim_{t\to a} \frac{1}{t-a}[\vec{r}(t) - \vec{r}(a)] = \lim_{h\to 0} \frac{1}{h}[\vec{r}(a+h) - \vec{r}(a)]$$

Then the equation of the tangent line to a curve traced by $\vec{r}(t)$ at the point P corresponding to the vector $\vec{r}(a) =< x(a), y(a) >$ is given by

$$\vec{L}(t) = \vec{r}(a) + t\vec{v}$$

Example 2. Find the tangent vector and parametric equations for the line tangent to the curve $\vec{r}(t) =< 1-4t, 2t-3t^2 >$ at the point $P(-11, -21)$.

Velocity.

Suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement of the object from the origin at time t. Function f is called the **position function** of the object.

$$\text{average velocity} \;=\; \frac{\text{displacement}}{\text{time}} = \frac{f(a+h)-f(a)}{h}$$

Then the **velocity** or **instantaneous velocity** at time $t = a$ is

$$v(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Example 3. The displacement of an object moving in a straight line is given by $s(t) = 1 + 2t + t^2/4$ (t is in seconds).

(a) Find the average velocity over the time period [1,3]

(b) Find the instantaneous velocity when $t = 1$

Example 4. The object is moving upward. Its height after t sec is given by $h(t) = 58t - 0.83t^2$

(a) What is the maximum height reached by the object?

(b) Find the instantaneous velocity at $t = 1$

Suppose an object moves in the xy-plane in such a way that its position at time t is given by the position vector $\vec{r}(t) =< x(t), y(t) >$.

$$\text{average velocity} = \frac{\vec{r}(a+h) - \vec{r}(h)}{h} = \frac{1}{h}[\vec{r}(a+h) - \vec{r}(a)]$$

The instantaneous velocity $\vec{v}(t)$ at the time $t = a$ is

$$\vec{v}(a) = \lim_{h \to 0} \frac{\vec{r}(a+h) - \vec{r}(a)}{h}$$

The **speed** of a particle is defined to be the magnitude of the velocity vector.

Example 5. If a ball is thrown into the air with a velocity of $10\vec{\imath} + 30\vec{\jmath}$ ft/s, its position after t seconds is given by $\vec{r}(t) = 10t\vec{\imath} + (30t - 16t^2)\vec{\jmath}$

(a) Find the velocity of the ball when $t = 1$

(b) Find the speed of the ball when $t = 1$

Other rates of change.

Suppose y is a quantity that depends on another quantity x or $y = f(x)$. If x changes from x_1 to x_2, then the change in x (also called the **increment** of x) is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

$$\Delta x = f(x_2) - f(x_1)$$

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the **average rate of change of y with respect to x** over the interval $[x_1, x_2]$.

The **instantaneous rate of change of y with respect to x** at $x = x_1$ is equal to

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$