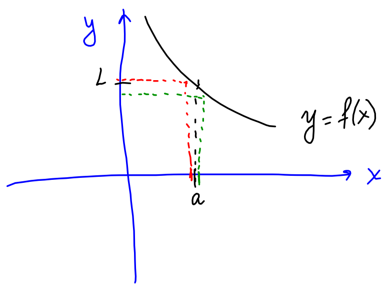
Chapter 2. **Limits and rates of change**
Section 2.2. **The limit of the function**

Definition. We write

$$\lim_{x \to a} f(x) = L$$

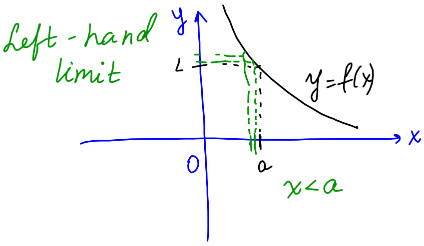
and say "the limit of $f(x)$, as x approaches a, equals L" if we can make values of $f(x)$ arbitrary close to L by taking x to be sufficiently close to a but not equal to a.

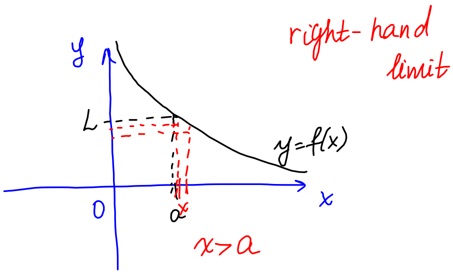


Definition. We write

$$\lim_{x \to a^-} f(x) = L$$

and say the left-handed limit **of $f(x)$ as x approaches a (or the limit of $f(x)$ as x approaches a from the left)**, equals L if we can make values of $f(x)$ arbitrary close to L by taking x to be sufficiently close to a and $x < a$.





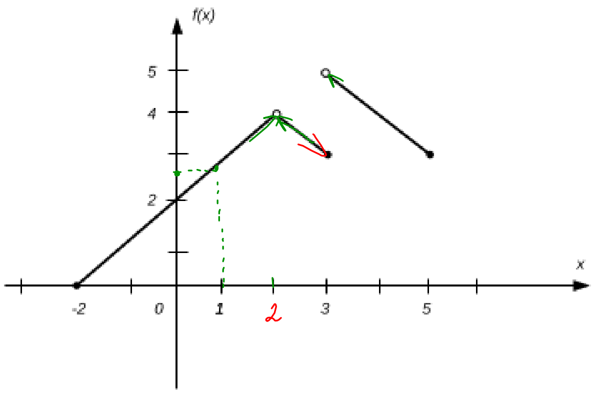
Definition. We write

$$\lim_{x \to a^+} f(x) = L$$

and say the **right-handed limit of $f(x)$ as x approaches a (or the limit of $f(x)$ as x approaches a from the right)**, equals L if we can make values of $f(x)$ arbitrary close to L by taking x to be sufficiently close to a and $x > a$.

$$\boxed{\lim_{x \to a} f(x) = L \text{ if and only if } \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L}$$

Example 1. Given the graph of the function f



Find:

1. $\lim_{x\to 1} f(x) = 2.5$
2. $\lim_{x\to 2^+} f(x) = 4$
3. $\lim_{x\to 2^-} f(x) = 4$

$\Big\}$ $\lim_{x\to 2} f(x) = 4$, $f(2)$ DNE

4. $\lim_{x\to 3^+} f(x) = 5$
5. $\lim_{x\to 3^-} f(x) = 3$

$\Big\}$ $\lim_{x\to 3} f(x)$ DNE

Definition. Let f be a function defined on both sides of a, except, possibly at a itself. Then

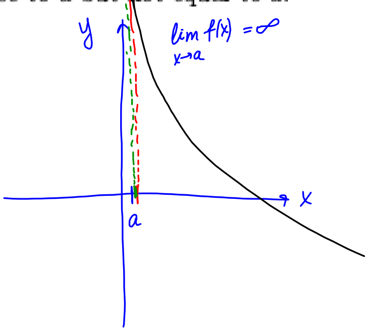
$$\lim_{x \to a} f(x) = \infty$$

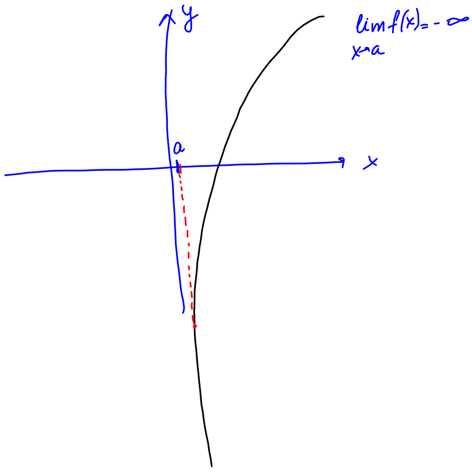
means that the values of $f(x)$ can be made arbitrary large by taking x to be sufficiently close to a but not equal to a.

Definition. Let f be a function defined on both sides of a, except, possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrary large negative by taking x to be sufficiently close to a but not equal to a.

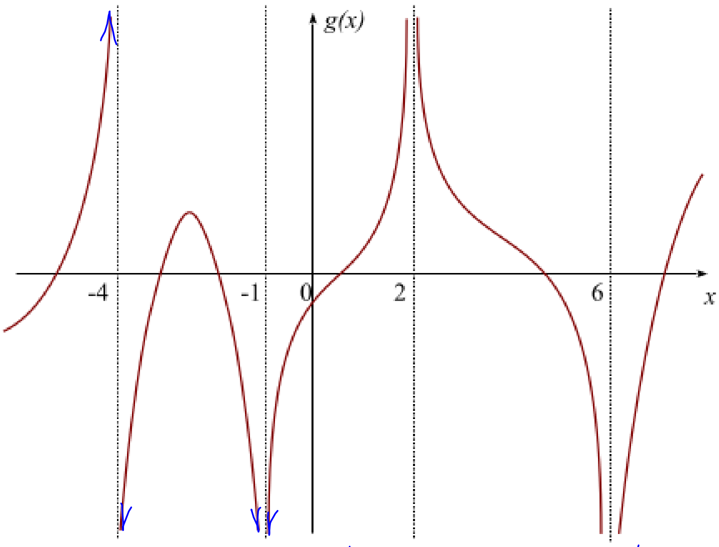




Definition. The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty$$
$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty$$

Example 2. For the function g whose graph is shown, state the following:



$\lim_{x\to -4^+} g(x) = -\infty$

$\lim_{x\to -4^-} g(x) = \infty$

1. $\lim_{x\to -4} g(x)$ $\boxed{\text{DNE}}$
2. $\lim_{x\to -1} g(x) = -\infty$
3. $\lim_{x\to 2} g(x) = \infty$
4. $\lim_{x\to 6} g(x) = -\infty$

Vertical asymptotes

$x = -4$

$x = -1$

$x = 2$

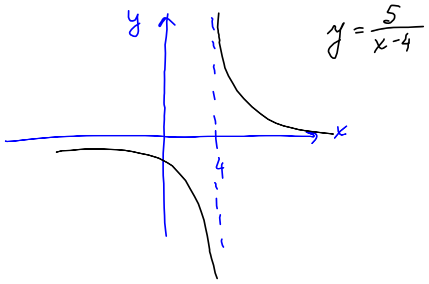
$x = 6$

Example 3. Find

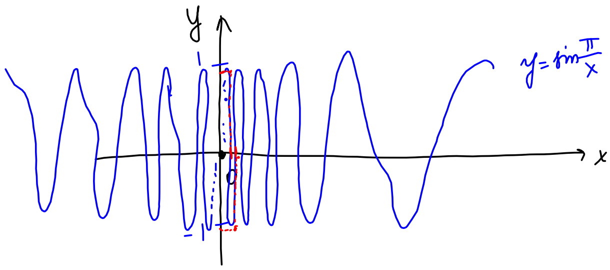
(a.) $\lim_{x\to 4^+} \frac{5}{x-4} = \infty$

(b.) $\lim_{x\to 4^-} \frac{5}{x-4} = -\infty$

(c.) $\lim_{x\to 4} \frac{5}{x-4}$ DNE



E. 4. Find $\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$ DNE, the values of f(x) oscillate between -1 and 1



Definition. We write

$$\lim_{t \to a} \vec{r}(t) = \vec{b}$$

and say "the limit of $\vec{r}(t)$, as t approaches a, equals $\vec{b}$" if we can make vector $\vec{r}(t)$ arbitrary close to $\vec{b}$ by taking t to be sufficiently close to a but not equal to a.

If $\vec{r}(t) =< f(t), g(t) >$, then

$$\boxed{\lim_{t \to a} \vec{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t) \right\rangle}$$

provided the limits of the component functions exist.