Chapter 2. **Limits and rates of change**
Section 2.3 **Calculating limits using the limit laws**

Limit laws Suppose that c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$
4. $\lim_{x\to a} f(x)g(x) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$
5. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
6. $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ where n is a positive integer
7. $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$
9. $\lim_{x\to a} x^n = a^n$ where n is a positive integer
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer
11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer

Example 1. Given that $\lim_{x\to a} f(x) = 2$, $\lim_{x\to a} g(x) = -1$, and $\lim_{x\to a} h(x) = 10$. Find the limits that exist.

1. $\lim_{x\to a}[2f(x) - g(x) - h(x)] = \lim_{x\to a} 2f(x) - \lim_{x\to a} g(x) - \lim_{x\to a} h(x)$

$$= 2\underbrace{\lim_{x\to a} f(x)}_{2} - \underbrace{\lim_{x\to a} g(x)}_{-1} - \underbrace{\lim_{x\to a} h(x)}_{10} = 2(2) - (-1) - 10 = \boxed{-5}$$

2. $\lim_{x\to a} \dfrac{g(x)}{h(x) - 2f(x)} = \dfrac{\lim_{x\to a} g(x)}{\lim_{x\to a} h(x) - 2\lim_{x\to a} f(x)} = \dfrac{-1}{10 - 2(2)} = \boxed{-\dfrac{1}{6}}$

Example 2. Evaluate the given limit and justify each step. DO NOT USE THE L'HOSPITAL'S RULE.

1. $\lim_{x \to 4} (2x^2 + 4x - 1) = 2(4)^2 + 4(4) - 1 = 2(16) + 16 - 1 = \boxed{47}$

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2. $\lim_{y \to 3} \frac{3(8y^2 - 1)}{2y^2(y-1)^4} = \frac{3(8(3)^2 - 1)}{2(3)^2(3-1)^2} = \frac{3(71)}{2(9)2^4} = \boxed{\frac{71}{96}}$

3. $\lim_{x \to 3} \sqrt[4]{x^2 + 2x + 1} = \sqrt[4]{(3)^2 + 2(3) + 1} = \sqrt[4]{16} = \boxed{2}$

If f is a polynomial or a rational function and a is in the domain of f, then $\lim_{x \to a} f(x) = f(a)$

Example 3. Evaluate each limit, if it exist.

1. $\lim_{x \to -1} \frac{x^2 - x - 2}{x + 1} = \left|\frac{0}{0}\right| = \lim_{x \to -1} \frac{\cancel{(x+1)}(x-2)}{\cancel{x+1}} = \lim_{x \to -1} (x-2) = \boxed{-3}$

2. $\lim_{x \to -1} \frac{x^2 - x - 3}{x + 1} = \left|\frac{-1}{0}\right|$ DNE

$x = -.99 \Rightarrow -102.... \leftarrow \lim_{x \to -1^+} \frac{x^2 - x - 3}{x+1} = -\infty$

$x = -1.01 \Rightarrow 96.99 \leftarrow \lim_{x \to -1^-} \frac{x^2 - x - 3}{x+1} = \infty$

4. $\lim_{t \to 1} \frac{t^3 - t}{t^2 - 1} = \lim_{t \to 1} \frac{t\cancel{(t^2-1)}}{\cancel{(t^2-1)}} = \lim_{t \to 1} t = \boxed{1}$

$(a+b)^2 = a^2 + 2ab + b^2$

5[redacted] $\lim_{h \to 0} \frac{(h+1)^2 - 1}{h} = \lim_{h \to 0} \frac{h^2 + 2h + \cancel{1} - \cancel{1}}{h} = \lim_{h \to 0} \frac{h^2 + 2h}{h}$

$= \lim_{h \to 0} \frac{\cancel{h}(h+2)}{\cancel{h}} = \lim_{h \to 0} (h+2) = \boxed{2}$

$(a-b)(a+b)=a^2-b^2$

6. $\displaystyle \lim_{x\to0}\frac{x}{\sqrt{1+3x}-1}\cdot\frac{(\sqrt{1+3x}+1)}{(\sqrt{1+3x}+1)} = \lim_{x\to0}\frac{x(\sqrt{1+3x}+1)}{(\sqrt{1+3x})^2-1^2} = \lim_{x\to0}\frac{x(\sqrt{1+3x}+1)}{1+3x-1}$

$\displaystyle = \lim_{x\to0}\frac{\cancel{x}(\sqrt{1+3x}+1)}{3\cancel{x}} = \lim_{x\to0}\frac{\sqrt{1+3x}+1}{3} = \boxed{\frac{2}{3}}$

7. $\displaystyle \lim_{x\to1}\left(\frac{1}{x-1}-\frac{2}{x^2-1}\right) = \lim_{x\to1}\frac{1(x+1)-2}{(x-1)(x+1)} = \lim_{x\to1}\frac{\cancel{x-1}}{\cancel{(x-1)}(x+1)} = \lim_{x\to1}\frac{1}{x+1} = \boxed{\frac{1}{2}}$

$x^2-1=(x-1)(x+1)$

8. $\displaystyle \lim_{t\to2}\vec{r}(t),\ \vec{r}(t)=\left\langle\frac{4-t}{2-\sqrt{t}},\frac{t^2-4}{t-2}\right\rangle$

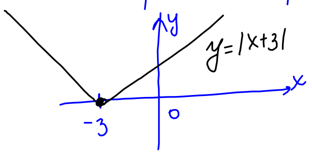
$\displaystyle \lim_{t\to2}\left\langle\frac{4-t}{2-\sqrt{t}},\frac{t^2-4}{t-2}\right\rangle = \left\langle\frac{4-2}{2-\sqrt{2}},\ \lim_{t\to2}\frac{\cancel{(t-2)}(t+2)}{\cancel{t-2}}\right\rangle$

$\displaystyle = \left\langle\frac{2}{2-\sqrt{2}},\ \lim_{t\to2}(t+2)\right\rangle = \boxed{\left\langle\frac{2}{2-\sqrt{2}},\ 4\right\rangle}$

9. $\displaystyle \lim_{x\to-3}|x+3| = \boxed{0}$

$|x+3| = \begin{cases} x+3, & \text{if } x\geq-3 \\ -(x+3), & \text{if } x<-3\end{cases}$

piecewise function



$\displaystyle \lim_{x\to-3^-}|x+3| = \lim_{x\to-3^-}[-(x+3)] = 0$

$(x<-3)$

match

$\displaystyle \lim_{x\to-3^+}|x+3| = \lim_{x\to-3^+}(x+3) = 0$

$(x>-3)$

10. $\displaystyle \lim_{x\to2}\frac{|x-2|}{x-2}$ $\boxed{\text{DNE}}$

$|x-2| = \begin{cases} x-2, & \text{if } x\geq2 \\ -(x-2), & \text{if } x<2\end{cases}$

$\displaystyle \lim_{x\to2^+}\frac{|x-2|}{x-2} = \lim_{x\to2^+}\frac{\cancel{x-2}}{\cancel{x-2}} = 1$

$(x>2)$

don't match

$\displaystyle \lim_{x\to2^-}\frac{|x-2|}{x-2} = \lim_{x\to2^-}\frac{-\cancel{(x-2)}}{\cancel{x-2}} = -1$

$(x<2)$

Example 4. Let

$$f(x) = \begin{cases} x^2 - 2x + 2, & \text{if } x < 1 \\ 3 - x, & \text{if } x \geq 1 \end{cases}$$

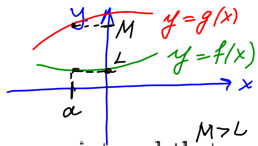
Find $\lim_{x\to 1} f(x)$. $\boxed{DNE}$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (3-x) = 3-1 = 2$ don't match

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2-2x+2) = 1-2+2 = 1$

Theorem If $f(x) \leq g(x)$ for all x in an open interval that contains a (except possibly at a) and the limits of f an g both exist as x approaches a, then

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$$



$L = \lim_{x\to a} f(x)$

$M = \lim_{x\to a} g(x)$

$M > L$

The Squeeze Theorem If $\overset{L \;(x\to a)}{f(x)} \leq g(x) \leq \overset{L \;(x\to a)}{h(x)}$ for all x in an open interval that contains a (except possibly at a) and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then

$$\lim_{x\to a} g(x) = L$$

Example 5. Use the Squeeze Theorem to show that $\lim_{x\to 0} x^2 \cos(20\pi x) = 0$.

$$\left(-1 \leq \cos(20\pi x) \leq 1\right)(x^2)$$

$$\underbrace{-x^2}_{f(x)} \leq x^2\cos(20\pi x) \leq \underbrace{x^2}_{h(x)}$$

$f(x) = -x^2$, $\lim_{x\to 0} (-x^2) = 0$

$h(x) = x^2$, $\lim_{x\to 0} x^2 = 0$

By the Squeeze Thm, $\lim_{x\to 0} x^2\cos(20\pi x) = 0$