Chapter 2. **Limits and rates of change**
Section 2.5 **Continuity**

Definition. A function f is **continuous at a number** a if

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$

$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$

If f is not continuous at a, then f has **discontinuity** at a:

- if $\boxed{\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)}$, then f has a **jump discontinuity** at a,
- if either $\boxed{\lim_{x \to a^+} f(x) = \infty \text{ or } \lim_{x \to a^-} f(x) = \infty}$ then f has an **infinity discontinuity** at a and we say line $x = a$ is a **vertical asymptote** of the curve $y = f(x)$
- if $\boxed{\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) \neq f(a)}$, then f has a **removable discontinuity** at a

hole in a graph

Example 1. Show that function $f(x) = x^2 + 2x + 3$ is continuous at $a = 2$.

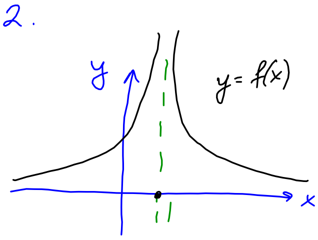
Show that $\lim_{x \to 2} f(x) = f(2)$

$\lim_{x \to 2} f(x) = \lim_{x \to 2} (x^2+2x+3) = 2^2+2(2)+3 = 11$

match

$f(2) = 2^2+2(2)+3 = 11$

Thus, $f(x) = x^2+2x+3$ is continuous @ 2.



Example 2. Explain why the function

$$f(x) = \begin{cases} \dfrac{1}{(x-1)^2}, & \text{if } x \neq 1 \\ 0, & \text{if } x = 1 \end{cases}$$

is discontinuous at $a = 1$. Sketch the graph of the function.

$\lim_{x \to 1} \frac{1}{(x-1)^2} = \infty$ infinity discontinuity @ 1.

Example 3. Find the points at which f is discontinuous.

$$f(x) = \begin{cases} \frac{1}{x}, & \text{if } x < -1 \quad 0 \text{ is not in the interval} \\ x, & \text{if } -1 \le x < 1 \\ \frac{1}{x^2}, & \text{if } x \ge 1 \quad 0 \text{ is not in the interval} \end{cases}$$

$\frac{1}{x}$ has the infinity discontinuity @ 0

$\frac{1}{x^2}$ has the infinity discontinuity @ 0

$x = -1$

$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} \frac{1}{x} = -1$ match

$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} x = -1$

$f(-1) = -1$

continuous @ -1

$x = 1$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1$ match

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{1}{x^2} = 1$

$f(1) = 1$

continuous @ 1

$\boxed{\text{NO POINTS OF DISCONTINUITY}}$

Definition. A function f is continuous from the right at a number a if

$$\lim_{x \to a^+} f(x) = f(a)$$

f is continuous from the left at a number a if

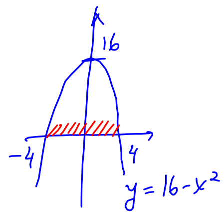
$$\lim_{x \to a^-} f(x) = f(a)$$

Definition A function f is continuous on an interval if it is continuous at every number in the interval. (At an endpoint of the interval we understand **continuous** to mean **continuous from the right** or **continuous from the left**.)

Example 4. Show that the function $f(x) = x\sqrt{16 - x^2}$ is continuous on its domain. State the domain.

$f(x) = x\sqrt{16-x^2}$. Domain: $16 - x^2 \geq 0$

$-4 \leq x \leq 4$



Pick an arbitrary point $-4 \leq t \leq 4$

Show that $f(x)$ is continuous @ t

$\lim_{x \to t} f(x) = f(t)$

$\lim_{x \to t} x\sqrt{16-x^2} = t\sqrt{16-t^2}$, $\quad f(t) = t\sqrt{16-t^2}$

match

thus, $f(x)$ is continuous on $[-4, 4]$

Continuous on $(-\infty, \infty)$

$f(x) = ax+b$

$f(x) = ax^2+bx+c$

. . . .

$f(x) = ax^n + bx^{n-1} + \ldots + cx + d$

$f(x) = \cos x$

$f(x) = \sin x$

Rational function

$$R(x) = \frac{ax^n + bx^{n-1} + \ldots + cx + d}{ex^m + fx^{m-1} + \ldots + gx + h}$$

continuous whenever

$ex^m + fx^{m-1} + \ldots + gx + h \neq 0$

$y = \tan x$

infinity discontinuities @

$x = \frac{\pi}{2} + \pi n, \quad n = 0, \pm 1, \pm 2, \ldots$

$y = \cot x$

infinity discontinuities @

$x = \pi n, \quad n = 0, \pm 1, \pm 2, \ldots$

Example 5. For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx+1, & \text{if } x \le 3 \\ cx^2 - 1, & \text{if } x > 3 \end{cases}$$

make sure that $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x)$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (cx+1) = 3c+1$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (cx^2 - 1) = 9c - 1$$

$3c + 1 = 9c - 1$

$9c - 3c = 1 + 1$

$6c = 2 \Rightarrow \quad c = \frac{2}{6} = \boxed{\frac{1}{3}}$

Theorem. If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f+g$ 2. $f-g$ 3. cf 4. fg 5. $\frac{f}{g}$ if $g(a) \neq 0$

Theorem.

(a.) Any polynomial is continuous on $(-\infty, \infty)$

(b.) Any rational function is continuous on its domain

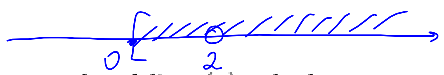
Theorem. If n is a positive even integer, then $f(x) = \sqrt[n]{x}$ is continuous on $[0, \infty)$. If n is a positive odd integer, then f is continuous on $(-\infty, \infty)$.

Example 6. On what interval is the function $h(x) = \sqrt{x} + \dfrac{1}{x-2} - \dfrac{1+2x}{x^2+4}$ continuous?

$\frac{1+2x}{x^2+4}$ continuous for all x.

$\frac{1}{x-2}$ infinity discontinuity @ 2

$\sqrt{x}$ continuous on $[0,\infty)$



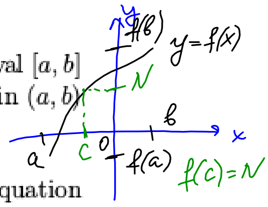
$\boxed{[0,2)\cup(2,\infty)}$

Theorem. If f is continuous at b and $\lim\limits_{x\to a} g(x) = b$, then

$$\lim_{x\to a} f(g(x)) = f(b) = f\left(\lim_{x\to a} g(x)\right)$$

Theorem. If g is continuous at a and f is continuous at $g(a)$, then $(f \circ g)(x) = f(g(x))$ is continuous at a.

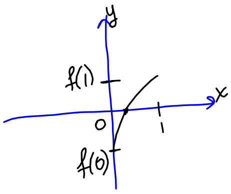
The intermediate value theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number strictly between $f(a)$ and $f(b)$. Then there exist a number c in (a, b) such that $f(c) = N$.



Example 7. Use the intermediate value theorem to show that there is a root of the equation $x^3 + 2x = x^2 + 1$ in the interval $(0,1)$.

$x^3+2x = x^2+1 \Rightarrow x^3+2x-x^2-1=0$

$f(x) = x^3+2x-x^2-1$



$f(0) = 0+0-0-1 = -1<0$

$f(1) = 1^3+2(1)-1^2-1 = 1>0$

By the Intermediate Value Theorem, there is a point $0<c<1$ such that $f(c)=0$