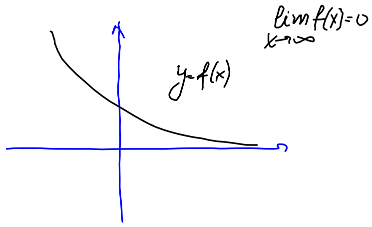


Section 2.6 Limits at infinity; horizontal asymptotes

Definition Let f be a function defined on (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

means that we can make values of $f(x)$ arbitrary close to L by taking x to be sufficiently large.

Definition Let f be a function defined on $(-\infty, a)$. Then

$$\lim_{x\to-\infty} f(x) = L$$

means that we can make values of $f(x)$ arbitrary close to L by taking x to be sufficiently large negative.

Definition The line $y = L$ is called a **horizontal asymptote of the curve $y = f(x)$** if either

$$\boxed{\lim_{x\to\infty} f(x) = L}$$

or

$$\boxed{\lim_{x\to-\infty} f(x) = L}$$

Limit laws Suppose that c is a constant and the limits $\lim_{x\to\pm\infty} f(x)$ and $\lim_{x\to\pm\infty} g(x)$ exist. Then

1. $\lim_{x\to\pm\infty} [f(x)+g(x)] = \lim_{x\to\pm\infty} f(x) + \lim_{x\to\pm\infty} g(x)$
2. $\lim_{x\to\pm\infty} [f(x)-g(x)] = \lim_{x\to\pm\infty} f(x) - \lim_{x\to\pm\infty} g(x)$
3. $\lim_{x\to\pm\infty} cf(x) = c \lim_{x\to\pm\infty} f(x)$
4. $\lim_{x\to\pm\infty} f(x)g(x) = \lim_{x\to\pm\infty} f(x) \cdot \lim_{x\to\pm\infty} g(x)$
5. $\lim_{x\to\pm\infty} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to\pm\infty} f(x)}{\lim_{x\to\pm\infty} g(x)}$ if $\lim_{x\to\pm\infty} g(x) \neq 0$
6. $\lim_{x\to\pm\infty} [f(x)]^n = \left[\lim_{x\to\pm\infty} f(x)\right]^n$ where n is a positive integer
7. $\lim_{x\to\pm\infty} c = c$
8. $\lim_{x\to\pm\infty} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to\pm\infty} f(x)}$ where n is a positive integer

Theorem If $r > 0$ is a rational number, then

$$\boxed{\lim_{x\to\infty} \frac{1}{x^r} = \lim_{x\to-\infty} \frac{1}{x^r} = 0}$$

Example 1. Find each of the following limits:

1

(a.) $\displaystyle \lim_{y\to\infty} \frac{7y^3+4y}{2y^3-y^2+3} = \lim_{y\to\infty} \frac{\cancel{y^3}\,\frac{7y^3+4y}{y^3}}{\cancel{y^3}\,\frac{2y^3-y^2+3}{y^3}} = \lim_{y\to\infty} \frac{\frac{7y^3}{y^3}+\frac{4y}{y^3}}{\frac{2y^3}{y^3}-\frac{y^2}{y^3}+\frac{3}{y^3}} = \lim_{y\to\infty} \frac{7+\cancel{\frac{4}{y^2}}^{\,0}}{2-\cancel{\frac{1}{y}}^{\,0}+\cancel{\frac{3}{y^3}}^{\,0}}$

$= \boxed{\frac{7}{2}}$

(b.) $\displaystyle \lim_{x\to\infty} \frac{x+4}{x^3-3} = \lim_{x\to\infty} \frac{\cancel{x}\,\frac{x+4}{x}}{x^{\cancel{3}2}\cdot\frac{x^3-3}{x^3}} = \lim_{x\to\infty} \frac{1}{x^2}\cdot\frac{\frac{x}{x}+\frac{4}{x}}{\frac{x^3}{x^3}-\frac{3}{x^3}} = \lim_{x\to\infty} \frac{1}{x^2}\cdot\frac{1+\cancel{\frac{4}{x}}^{\,0}}{1-\cancel{\frac{3}{x^3}}^{\,0}}$

$\displaystyle = \lim_{x\to\infty} \frac{1}{x^2} = \boxed{0}$

(c.) $\displaystyle \lim_{t\to-\infty} \frac{t^2-3t+1}{2t+3} = \lim_{t\to-\infty} \frac{t^{\cancel{2}}\,\frac{t^2-3t+1}{t^2}}{\cancel{t}\,\frac{2t+3}{t}} = \lim_{t\to-\infty} \frac{t\cdot\left(1-\cancel{\frac{3}{t}}^{\,0}+\cancel{\frac{1}{t^2}}^{\,0}\right)}{2+\cancel{\frac{3}{t}}^{\,0}} = \lim_{t\to-\infty} \frac{t}{2} = \boxed{-\infty}$

$\displaystyle \lim_{t\to-\infty} \frac{t^5-5t+151}{t^3+2t+2016} = \lim_{t\to-\infty} \frac{t^5\left(1-\cancel{\frac{5}{t^4}}^{\,0}+\cancel{\frac{151}{t^5}}^{\,0}\right)}{t^3\left(1+\cancel{\frac{2}{t^2}}^{\,0}+\cancel{\frac{2016}{t^3}}^{\,0}\right)} = \lim_{t\to-\infty} t^2 = \boxed{\infty}$

$$\lim_{x\to\infty} \frac{\boxed{a_n}x^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0}{\boxed{b_m}x^m+b_{m-1}x^{m-1}+\ldots+b_1x+b_0} = \begin{cases} \frac{a_n}{b_m}, & \text{if } n=m \\ 0, & \text{if } n<m \\ \infty, & \text{if } n>m \text{ and } \frac{a_n}{b_m}>0 \\ -\infty, & \text{if } n>m \text{ and } \frac{a_n}{b_m}<0 \end{cases}$$

∞, if $n > m$

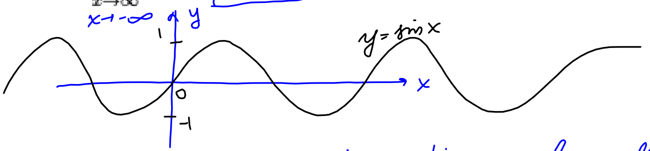
Example 2. Evaluate the following limits:

(a.) $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2+4x}}{4x+1} = \lim\limits_{x\to\infty} \dfrac{\sqrt{x^2 \frac{x^2+4x}{x^2}}}{x\,\frac{4x+1}{x}} = \lim\limits_{x\to\infty} \dfrac{\cancel{x}\sqrt{1+\frac{4}{x}}}{\cancel{x}\left(4+\frac{1}{x}\right)} = \boxed{\dfrac{1}{4}}$

$\lim\limits_{x\to+\infty} \cos x \; \boxed{\text{DNE}}$

(b.) $\lim\limits_{x\to\infty} \sin x \; \boxed{\text{DNE}}$

$x\to-\infty$



as $x\to\infty$, values of $\sin x$ oscillate between -1 and 1, they do not approach to a number

$\sin x$ is continuous for all x.

(c.) $\lim\limits_{x\to\infty} \sin\dfrac{1}{x}$

$= \sin\left(\lim\limits_{x\to\infty} \dfrac{1}{x}\right) = \sin 0 = \boxed{0}$

(d.) $\lim\limits_{x\to\infty} \dfrac{\sin^2 x}{x^2}$

Squeeze Thm.

$\dfrac{0}{x^2} \le \dfrac{\sin^2 x}{x^2} \le \dfrac{1}{x^2}$

$0 \le \dfrac{\sin^2 x}{x^2} \le \dfrac{1}{x^2}$

$\lim\limits_{x\to\infty} 0 = 0$

$\lim\limits_{x\to\infty} \dfrac{1}{x^2} = 0$

By the Squeeze Thm,

$\lim\limits_{x\to\infty} \dfrac{\sin^2 x}{x^2} = \boxed{0}$

$\lim\limits_{x\to\infty} \dfrac{\sin^2 x}{x^2} = 0$

$1-\cos x = 2\sin^2\dfrac{x}{2}$

(e.) $\lim\limits_{x\to\infty} \dfrac{1-\cos x}{x^2} = \lim\limits_{x\to\infty} \dfrac{2\sin^2\frac{x}{2}}{x^2}$

substitution $\dfrac{x}{2} = t$, $t\to\infty$ as $x\to\infty$

$x = 2t$

$x^2 = 4t^2$

$= \lim\limits_{t\to\infty} \dfrac{2\sin^2 t}{4t^2} = \dfrac{1}{2} \cancel{\lim\limits_{t\to\infty} \dfrac{\sin^2 t}{t^2}}^{\,0} = \boxed{0}$

(f.) $\lim\limits_{x\to\infty} \left(\sqrt{x^2+3x+1} - x\right) \dfrac{\left(\sqrt{x^2+3x+1}+x\right)}{\sqrt{x^2+3x+1}+x} = \lim\limits_{x\to\infty} \dfrac{\left(\sqrt{x^2+3x+1}\right)^2 - x^2}{\sqrt{x^2+3x+1}+x}$

$(a-b)(a+b) = a^2 - b^2$

$= \lim\limits_{x\to\infty} \dfrac{\cancel{x^2}+3x+1-\cancel{x^2}}{\sqrt{x^2+3x+1}+x} = \lim\limits_{x\to\infty} \dfrac{3x+1}{\sqrt{x^2\left(1+\cancel{\frac{3}{x}}+\cancel{\frac{1}{x^2}}\right)}+x}$

$= \lim\limits_{x\to\infty} \dfrac{3x+1}{2x} = \boxed{\dfrac{3}{2}}$

Example 3. Find the horizontal and vertical asymptotes of each curve

(a.) $y = \dfrac{x^2+4}{x^2-1}$

V.A. $x^2-1=0$

$\boxed{x=1}$ or $\boxed{x=-1}$

H.A. $\lim_{x\to\infty} \dfrac{x^2+4}{x^2-1} = \lim_{x\to\infty} \dfrac{\cancel{x^2}(1+\frac{4}{x^2})}{\cancel{x^2}(1-\frac{1}{x^2})} = 1$

$\boxed{y=1}$

(b.) $y = \dfrac{x^3+1}{x^2+x}$

$\lim_{x\to\infty} \dfrac{x^3+1}{x^2+x} = \infty$ no horizontal asymptotes

$\boxed{\begin{array}{l} a^3+b^3=(a+b)(a^2-ab+b^2) \\ a=x,\ b=1 \end{array}}$

$\dfrac{x^3+1}{x^2+x} = \dfrac{\cancel{(x+1)}(x^2-x+1)}{x\cancel{(x+1)}} = \dfrac{x^2-x+1}{x}$

removable discontinuity @ $x=-1$

V.A. $\boxed{x=0}$