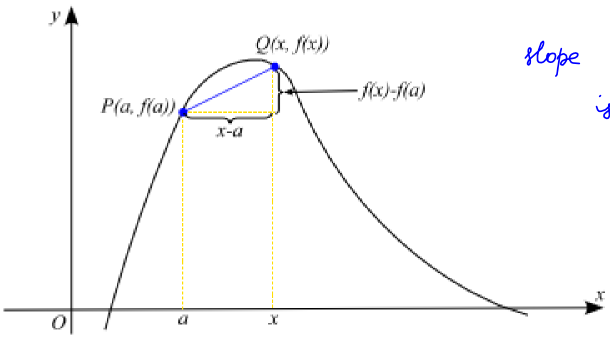
Section 2.7 **Tangents, velocities, and other rates of change**

The tangent line.

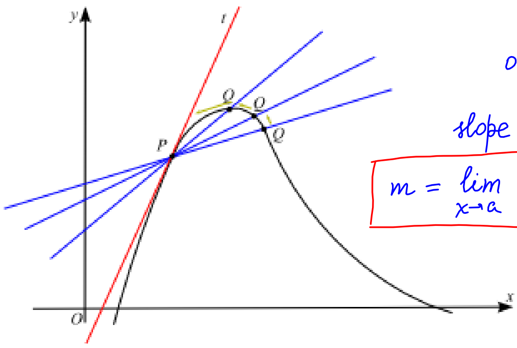
If a curve C has equation $y = f(x)$ and we want to find the tangent to C at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line PQ:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$



slope of the secant line is $\frac{f(x)-f(a)}{x-a}$

Then we let Q approach P along the curve C by letting x approach a.



push Q toward P or $x \to a$

slope of the tangent line is

$$m = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} = f'(a)$$

If m_{PQ} approaches a number m, then we define the **tangent** t to be the line through P with slope m.

Definition. The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

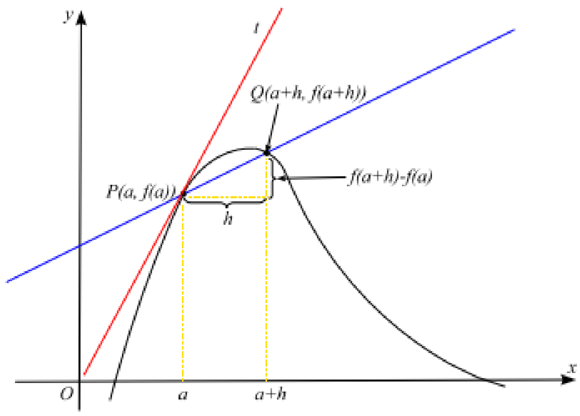
provided that this limit exists. Then the equation of the tangent line is

$$y = m(x - a) + f(a) \quad \text{or} \quad y = f'(a)(x-a) + f(a)$$

$h = x - a$

Let h [redacted] then $x = a + h$, so the slope of the secant line PQ is

$$m_{PQ} = \frac{f(a+h) - f(a)}{h}$$



Then the slope of the tangent line becomes

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

Example 1. Find the equation of the tangent line to the curve $y = \sqrt{2x - 3}$ at the point $(2, 1)$. ($a = 2$, $f(a) = 1$)

$$\text{slope} = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}, \quad f(x) = \sqrt{2x-3}, \quad f(2) = 1$$

$$= \lim_{x \to 2} \frac{(\sqrt{2x-3} - 1)(\sqrt{2x-3} + 1)}{(x-2)(\sqrt{2x-3} + 1)} = \lim_{x \to 2} \frac{(\sqrt{2x-3})^2 - 1}{(x-2)(\sqrt{2x-3}+1)} = \lim_{x \to 2} \frac{2x-3-1}{(x-2)(\sqrt{2x-3}+1)}$$

$$= \lim_{x \to 2} \frac{2x-4}{(x-2)(\sqrt{2x-3}+1)} = \lim_{x \to 2} \frac{2(x-2)}{(x-2)(\sqrt{2x-3}+1)} = \frac{2}{\sqrt{2(2)-3}+1} = \frac{2}{2} = \boxed{1}$$

Equation of the tangent line:

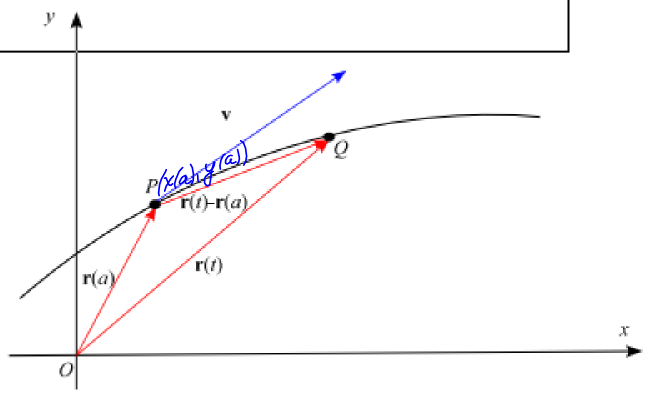
$$y = 1(x-2) + 1$$

$$\boxed{y = x - 1}$$

Tangent vectors

Let $\vec{r}(t) =< x(t), y(t) >$ be a vector function.

Problem. Find a tangent vector to a curve traced by $\vec{r}(t)$ at the point P corresponding to the vector $\vec{r}(a) =< x(a), y(a) >$.



The tangent vector to a curve traced by $\vec{r}(t)$ at the point P corresponding to the vector $\vec{r}(a) =< x(a), y(a) >$ is given by

$$\text{tangent vector } \vec{v} = \lim_{t\to a} \frac{1}{t-a}[\vec{r}(t) - \vec{r}(a)] = \lim_{h\to 0} \frac{1}{h}[\vec{r}(a+h) - \vec{r}(a)] = r'(a) = < x'(a), y'(a) >$$

Then the equation of the tangent line to a curve traced by $\vec{r}(t)$ at the point P corresponding to the vector $\vec{r}(a) =< x(a), y(a) >$ is given by

$$\vec{L}(t) = \vec{r}(a) + t\vec{v}$$

Example 2. Find the tangent vector and parametric equations for the line tangent to the curve $\vec{r}(t) =< 1 - 4t, 2t - 3t^2 >$ at the point $P(-11, -21)$.

find t such that $1-4t=-11$ and $2t-3t^2=-21$

$-4t=-12$

$t=3 \Rightarrow <-11,-21> = \vec{r}(3)$

tangent vector $\vec{v} = \lim_{t\to 3} \frac{\vec{r}(t)-\vec{r}(3)}{t-3} = \lim_{t\to 3} \frac{1}{t-3}\left[<1-4t, 2t-3t^2> - <-11,-21>\right]$

$= \lim_{t\to 3} < \frac{1-4t+11}{t-3}, \frac{2t-3t^2+21}{t-3} > = < \lim_{t\to 3} \frac{12-4t}{t-3}, \lim_{t\to 3} \frac{2t-3t^2+21}{t-3} >$

$= < \lim_{t\to 3} \frac{-4(t-3)}{t-3}, \lim_{t\to 3} \frac{(t-3)(-7-3t)}{t-3} > = <-4,-16>$

$\vec{v}(t) = \vec{r}\,'(t) = <-4, 2-6t>$

$\vec{v}(3) = <-4,-16>$

Equations of the tangent line:

vector: $<x(t), y(t)> = <-11,-21> + t<-4,-16>$

parametric: $x = -11-4t$, $y = -21-16t$

Suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement of the object from the origin at time t. Function f is called the **position function** of the object.

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{h}$$

Then the **velocity** or **instantaneous velocity** at time $t = a$ is

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Example 3. The displacement of an object moving in a straight line is given by $s(t) = 1 + 2t + t^2/4$ (t is in seconds).

(a) Find the average velocity over the time period [1,3] $= f'(a)$

(b) Find the instantaneous velocity when $t = 1$

$$V_{ave} = \frac{S(3)-S(1)}{3-1} = \frac{1+2(3)+\frac{3^2}{4}-\left(1+2(1)+\frac{1}{4}\right)}{2} = \frac{7+\frac{9}{4}-3-\frac{1}{4}}{2} = \frac{4+\frac{8}{4}}{2} = \boxed{3}$$

$$V(t) = S'(t) = 2+\frac{2t}{4} = 2+\frac{t}{2}$$

$$V(1) = 2+\frac{1}{2} = \boxed{\frac{3}{2}}$$

Example 4. The object is moving upward. Its height after t sec is given by $h(t) = 58t - 0.83t^2$

(a) What is the maximum height reached by the object?

max height is reached when $V(t) = 0$.

$$V(t) = \lim_{s \to 0} \frac{h(s+t)-h(t)}{s} = (58t - 0.83t^2)' = 58 - 2(0.83)t = 58 - 1.66t = 0$$

$$t = \frac{58}{1.66} \approx 35 \text{ (sec)}, \quad h(35) = 58(35) - 0.83(35)^2 = \boxed{1013.25}$$

(b) Find the instantaneous velocity at $t = 1$

$$V(1) = 58 - 1.66 = \boxed{56.34}$$

Suppose an object moves in the xy-plane in such a way that its position at time t is given by the position vector $\vec{r}(t) =< x(t), y(t) >$.

$$\text{average velocity} = \frac{\vec{r}(a+h) - \vec{r}(h)}{h} = \frac{1}{h}[\vec{r}(a+h) - \vec{r}(a)]$$

The instantaneous velocity $\vec{v}(t)$ at the time $t = a$ is

$$\vec{v}(a) = \lim_{h\to 0} \frac{\vec{r}(a+h) - \vec{r}(a)}{h} = \vec{r}\,'(a) = <x'(a), y'(a)>$$

The speed of a particle is defined to be the magnitude of the velocity vector.

Example 5. If a ball is thrown into the air with a velocity of $10\vec{\imath} + 30\vec{\jmath}$ ft/s, its position after t seconds is given by $\vec{r}(t) = 10t\vec{\imath} + (30t - 16t^2)\vec{\jmath}$

(a) Find the velocity of the ball when $t = 1$

$\vec{v}(0) = <10, 30>$, $\quad \vec{r}(t) = <10t, 30t - 16t^2>$

velocity vector $\vec{v}(t) = <(10t)', (30t - 16t^2)'>$

$= <10, 30 - 16(2t)> = <10, 30 - 32t>$

$\vec{v}(1) = <10, 30 - 32(1)> = \boxed{<10, -2>}$

(b) Find the speed of the ball when $t = 1$

speed $= |\vec{v}(1)| = \sqrt{10^2 + (-2)^2} = \boxed{\sqrt{104}}$ (ft/s)

Other rates of change.

Suppose y is a quantity that depends on another quantity x or $y = f(x)$. If x changes from x_1 to x_2, then the change in x (also called the **increment** of x) is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1)$$

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

average rate of change

is called the **average rate of change of y with respect to x** over the interval $[x_1, x_2]$.

The **instantaneous rate of change of y with respect to x at $x = x_1$** is equal to

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_1)$$