Section 3.11 **Differentials; linear and quadratic approximations**

Definition Let $y = f(x)$, where f is a differentiable function. Then the **differential** dx is an independent variable; that is dx can be given the value of any real number. The **differential** dy is then defined in terms of dx by the equation

$$dy = f'(x)dx$$

Example 1. Find dy if $y = x \tan x$.

Example 2.

(a.) Find dy if $y = \sqrt{1-x}$

(b.) Find the value of dy when $x = 0$ and $dx = .02$

Suppose that $f(a)$ is a known number and the approximate value is to be calculated for $f(a + \Delta x)$ where Δx is small. Then

$$f(a + \Delta x) \approx f(a) + dy = f(a) + f'(a)\Delta x$$

Example 3. Use differentials to find an approximate value for

(a.) $\sqrt{36.1}$

(b.) $\sin 59^0$

Linear approximations.

The approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the **linear approximation** or **tangent line approximation** of f at a, and the function

$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of f at a.

Example 4. Find the linearization $L(x)$ of the function $f(x) = \dfrac{1}{\sqrt{2+x}}$ at $a = 0$

Example 5. Verify the linear approximation at $a = 0$

(a.) $\sqrt{1+x} \approx 1 + \frac{1}{2}x$

(b.) $\sin x \approx x$

Example 6. Find the linear approximation of the function $f(x) = \sqrt{1-x}$ at $a = 0$ and use it to approximate the numbers $\sqrt{0.9}$ and $\sqrt{0.99}$.

Quadratic approximations

Let's approximate curve $y = f(x)$ by a parabola $y = P(x)$ near $x = a$. To make sure that the approximation is a good one, we stipulate the following:

$$P(a) = f(a)$$

$$P'(a) = f'(a)$$

$$P''(a) = f''(a)$$

Example 7. Find the quadratic approximation for the function $f(x) = \cos x$ near $x = 0$.

The **quadratic approximation** of f near a is

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

Example 8. Find the quadratic approximation to $f(x) = \sqrt[3]{x}$ near $a = -8$.