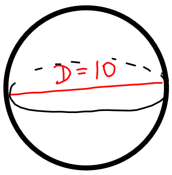
Section 3.10 **Related rates**

Strategy

1. Read the problem carefully.
2. Draw a diagram if possible.
3. Introduce notation. Assign symbols to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives.
5. Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution.
6. Use the Chain Rule to differentiate both sides of the equation with respect to t.
7. Substitute the given information into the resulting equation and solve for the unknown rate.

Example 1. A spherical snowball is melting in such a way that its volume is decreasing at a rate of 1 cm^3/min. At what rate is the diameter decreasing when the diameter is 10 cm.



D is the diameter

V is the volume

$\frac{dV}{dt}$ is the rate of increasing/decreasing of the volume

$\frac{dD}{dt}$ is the rate of increasing/decreasing of the diameter

$D = 10, \quad \frac{dV}{dt} = -1$

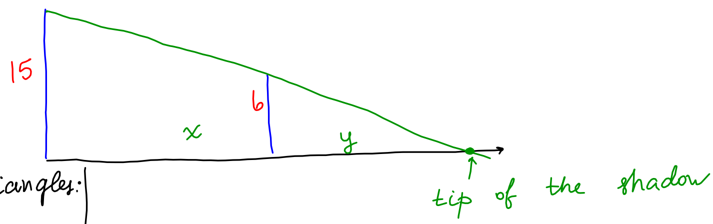
$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{D}{2}\right)^3 = \frac{4}{3}\pi\frac{D^3}{8} = \boxed{\frac{\pi}{6}D^3 = V}$$

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{\pi}{6}D^3\right)$$
$$= \frac{\pi}{6}\frac{d}{dt}\left(D^3\right)$$
$$= \frac{\pi}{6}\, 3D^2 \frac{dD}{dt}$$
$$\frac{dV}{dt} = \frac{\pi}{2}D^2\frac{dD}{dt}$$
$$\frac{dD}{dt} = \frac{dV}{dt}\cdot\frac{2}{\pi D^2} = (-1)\cdot\frac{2}{\pi(10^2)} = \frac{-2}{100\pi} = -\frac{1}{50\pi}$$

$\boxed{\text{the diameter decreases at a rate of } \frac{1}{50\pi} \text{ cm/min}}$

Example 2. A street light is at the top of a 15-ft-tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path.

(a.) How fast is the tip of his shadow moving when he is 40 ft from the pole?



x is the distance between the man and the pole

y is the length of the shadow

$\frac{dx}{dt}$ speed of the man

$\frac{dy}{dt}$ rate of lengthens of the shadow

similar triangles:

$$\frac{15}{6} = \frac{x+y}{y}$$

$$\frac{5}{2} = \frac{x+y}{y}$$

$$5y = 2(x+y)$$

$$5y = 2x + 2y$$

$$3y = 2x$$

$$y = \frac{2}{3}x$$

$x+y$ is the distance from the tip of the shadow to the pole

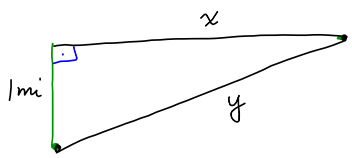
$\frac{d(x+y)}{dt}$ is the speed of the tip of the shadow

$$\frac{d(x+y)}{dt} = \frac{d\left(x + \overbrace{\tfrac{2}{3}x}^{y}\right)}{dt} = \frac{d\left(\frac{5}{3}x\right)}{dt} = \frac{5}{3}\overbrace{\frac{dx}{dt}}^{5} = \boxed{\frac{25}{3}\ ft/s}$$

(b.) How fast is his shadow lengthening at that point?

Find $\frac{dy}{dt} = \frac{d\left(\frac{2}{3}x\right)}{dt} = \frac{2}{3}\frac{dx}{dt} = \frac{2}{3}(5) = \boxed{\frac{10}{3}\ (ft/s)}$

Example 3. A plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 2 mi away from the station.

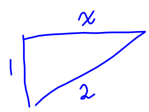


$\frac{dx}{dt}$ the speed of the plane

$\frac{dx}{dt} = 500$

y is the distance from the plane to the station.

Find $\frac{dy}{dt}$ when $y = 2$



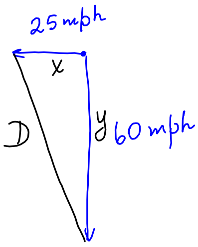
$\frac{d}{dt}(x^2 + 1) = \frac{d}{dt}(y^2)$

$\cancel{2}x \frac{dx}{dt} = \cancel{2}y \frac{dy}{dt} \longrightarrow \frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$

$x = \sqrt{4-1} = \sqrt{3}$ when $y = 2$.

$\frac{dy}{dt} = \frac{\sqrt{3}}{2}(500) = \boxed{250\sqrt{3} \text{ mph}}$

Example 4. Two cars start moving from the same point. One travels south at 60 mi/h and the other travels west at 25 mi/h. At what rate is the distance between the cars increasing two hours later?



$\frac{dx}{dt} = 25, \quad \frac{dy}{dt} = 60$

$x = (25)(2) = 50$

$y = (60)(2) = 120$

$D = \sqrt{x^2+y^2} = \sqrt{50^2+120^2} = 130$

D is the distance between the cars

x is the distance traveled by car 2

y is the distance traveled by car 1

$\frac{dx}{dt}$ is the speed of car 2

$\frac{dy}{dt}$ is the speed of car 1

$\frac{dD}{dt}$ rate of increase/decrease of the distance between the cars.

$\frac{d}{dt}(D^2) = \frac{d}{dt}(x^2+y^2)$

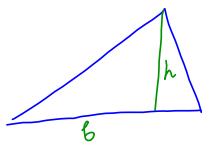
$\cancel{2}D \frac{dD}{dt} = \cancel{2}x \frac{dx}{dt} + \cancel{2}y \frac{dy}{dt}$

$\frac{dD}{dt} = \frac{1}{D}\left(x \frac{dx}{dt} + y \frac{dy}{dt}\right)$

$= \frac{1}{130}\left(50(25) + 120(60)\right) = \boxed{65 \text{ mph}}$

• HW over 3.8 – 3.10 is due 10/19, 11:55 PM
• 10/17 OFFICE HOURS MOVED TO 11:30–1:30 PM.

Example 5. The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm^2/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2?



$\frac{dh}{dt} = 1$, $\frac{dA}{dt} = 2$

Find $\frac{db}{dt}$, when $h = 10$, $A = 100$

$A = \frac{1}{2} b h$

$100 = \frac{1}{2} b (10)$

$b = \frac{100}{5} = 20$

h is the altitude

b is the base

A is the area

$\frac{dh}{dt}$ is rate of change of the altitude

$\frac{db}{dt}$ is the rate of change of the base

$\frac{dA}{dt}$ is the rate of change of the area.

$$\frac{d}{dt}(A) = \frac{d}{dt}\left(\frac{1}{2} b h\right)$$

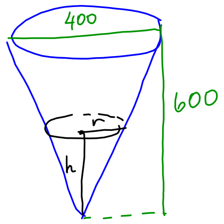
$$\frac{dA}{dt} = \frac{1}{2}\left(\frac{db}{dt} h + b \frac{dh}{dt}\right)$$

$$\frac{1}{2} \frac{db}{dt} h = \frac{dA}{dt} - \frac{b}{2} \frac{dh}{dt}$$

$$\frac{db}{dt} = \frac{2}{h}\left(\frac{dA}{dt} - \frac{b}{2} \frac{dh}{dt}\right)$$

$$= \frac{2}{10}\left(2 - \frac{20}{2}(1)\right) = \frac{1}{5}(-8) = \boxed{-\frac{8}{5} \text{ cm/min}}$$

Example 6. Water is leaking out of an inverted conical tank at a rate of 10,000 cm^3/min at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 2m, find the rate at which water is being pumped into the tank.

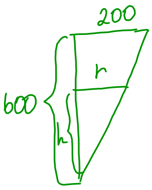


h is the height of the water in the tank

$\frac{dh}{dt} = 20$

r is the radius of the cone at the height h.

V is the amount of the water in the tank



Similar triangles:

$\frac{600}{200} = \frac{h}{r}$

$r = \frac{h}{3}$

$h = 2\,m = 200\,cm$

$\frac{dV}{dt} = [\text{rate in}] - [\text{rate out}]$

x is the rate at which water is being pumped into the tank

$\frac{dV}{dt} = x - 10{,}000$. Find x.

$x = \frac{dV}{dt} + 10{,}000$

$V = \frac{1}{3}\pi r^2 h$. Eliminate $r = \frac{h}{3}$

$\frac{dV}{dt} = \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h = \frac{\pi}{27}\frac{d}{dt}(h^3)$

$\frac{dV}{dt} = \frac{\pi}{27}3h^2\frac{dh}{dt} = \frac{\pi h^2}{9}\frac{dh}{dt}$

$= \frac{\pi(200)^2}{9}(20) = \frac{\pi(800{,}000)}{9}$

$\boxed{x = \frac{\pi(800{,}000)}{9} + 10{,}000}$ cm^3/min