Section 3.11 **Differentials; linear and quadratic approximations**

Definition Let $y = f(x)$, where f is a differentiable function. Then the **differential** dx is an independent variable; that is dx can be given the value of any real number. The differential dy is then defined in terms of dx by the equation

$$\boxed{dy = f'(x)dx}$$

Example 1. Find dy if $y = x \tan x$.

Product Rule

$$dy = (x \tan x)' \, dx$$

$$\boxed{dy = (\tan x + x \sec^2 x)\, dx}$$

Example 2.

(a.) Find dy if $y = \sqrt{1-x} = (1-x)^{1/2}$

$$dy = [(1-x)^{1/2}]' \, dx$$

$$= \frac{1}{2}(1-x)^{-1/2}(1-x)' \, dx$$

$$= \boxed{-\frac{1}{2}(1-x)^{-1/2} \, dx}$$

(b.) Find the value of dy when $x = 0$ and $dx = .02$

$$dy = -\frac{1}{2}(1-0)^{-1/2}(.02)$$

$$= \boxed{-0.01}$$

Suppose that $f(a)$ is a known number and the approximate value is to be calculated for $f(a+\Delta x)$ where Δx is small. Then

$$f(a+\Delta x) \approx f(a) + dy = \boxed{f(a) + f'(a)\Delta x \approx f(a+\Delta x)}$$

Example 3. Use differentials to find an approximate value for

(a.) $\sqrt{36.1}$ $= \sqrt{36+.1}$ $\quad a=36, \ \Delta x = .1, \ f(x)=\sqrt{x}$

$f(x)=\sqrt{x}$ $\quad f(36)=6$

$f'(x)=\frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ $\quad f'(36)=\frac{1}{2\sqrt{36}}=\frac{1}{12}$

$\sqrt{36.1} \approx f(36) + f'(36)\Delta x$

$\sqrt{36.1} \approx 6 + \frac{1}{12}(0.1) \approx 6.008$

(b.) $\sin 59^0$ $= \sin(60^\circ - 1^\circ)$

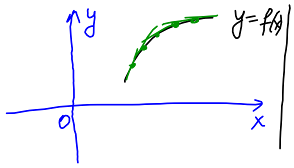
convert degrees into radians.

$a = 60^\circ = \frac{\pi}{3}, \quad \Delta x = -1^\circ = -\frac{\pi}{180}$

$f(x) = \sin x, \quad f\left(\frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$f'(x) = \cos x, \quad f'\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$

$\sin 59^\circ \approx \sin\frac{\pi}{3} + \cos\frac{\pi}{3}\left(-\frac{\pi}{180}\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}\frac{\pi}{180} = \boxed{\frac{180\sqrt{3}-\pi}{360}} \approx .857$



~~Approximating~~ Approximating a curve $y=f(x)$ by tangent lines.

Linear approximations.

The approximation

$$\boxed{f(x) \approx f(a) + f'(a)(x-a)}$$

is called the linear approximation or **tangent line approximation** of f at a, and the function

$$\boxed{L(x) = f(a) + f'(a)(x-a)}$$

is called the linearization of f at a.

Example 4. Find the linearization $L(x)$ of the function $f(x) = \dfrac{1}{\sqrt{2+x}}$ at $a = 0$

$L(x) = f(0) + f'(0)(x-0)$, $\quad f(x) = (2+x)^{-1/2}$, $\quad f(0) = \frac{1}{\sqrt{2}}$

$f'(x) = -\frac{1}{2}(2+x)^{-3/2}$, $\quad f'(0) = -\frac{1}{2}\, 2^{-3/2}$

$= -\frac{1}{2} \cdot \frac{1}{2\sqrt{2}}$

$= -\frac{1}{4\sqrt{2}}$

$$\boxed{L(x) = \frac{1}{\sqrt{2}} - \frac{1}{4\sqrt{2}}x}$$

Example 5. Verify the linear approximation at $a = 0$

(a.) $\sqrt{1+x} \approx 1 + \frac{1}{2}x$

$f(x) = \sqrt{1+x}$, $\quad f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-1/2}$, $\quad f'(0) = \frac{1}{2}$

$\sqrt{1+x} \approx f(0) + f'(0)(x-0)$

$$\boxed{\sqrt{1+x} \approx 1 + \frac{1}{2}x}$$

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(b.) $\sin x \approx x$

$f(x) = \sin x$, $\quad f(0) = 0$

$f'(x) = \cos x$, $\quad f'(0) = 1$

$f(x) = \sin x \approx f(0) + f'(0)x$

$\sin x \approx 0 + x$

$$\boxed{\sin x \approx x}$$

Example 6. Find the linear approximation of the function $f(x) = \sqrt{1-x}$ at $a = 0$ and use it to approximate the numbers $\sqrt{0.9}$ and $\sqrt{0.99}$.

$f(x) = (1-x)^{1/2}$, $f(0) = 1$

$f'(x) = \frac{1}{2}(1-x)^{-1/2}\,(1-x)'$

$= -\frac{1}{2}(1-x)^{-1/2}$, $f'(0) = -\frac{1}{2}$

$\sqrt{1-x} \approx f(0) + f'(0)x$

$\boxed{\sqrt{1-x} \approx 1 - \frac{1}{2}x}$

$\sqrt{0.9} = \sqrt{1-0.1} \approx 1 - \frac{1}{2}(0.1) = 1 - 0.05 = \boxed{0.95}$

$\sqrt{0.99} = \sqrt{1-0.01} \approx 1 - \frac{1}{2}(0.01) = \boxed{.995}$

Quadratic approximations

$y = P(x) = k(x-a)^2 + m(x-a) + \ell$

k, m, ℓ are unknowns

Let's approximate curve $y = f(x)$ by a parabola $y = P(x)$ near $x = a$. To make sure that the approximation is a good one, we stipulate the following:

$$P(a) = f(a)$$

$$P'(a) = f'(a)$$

$$P''(a) = f''(a)$$

$P(x) = k(x-a)^2 + m(x-a) + \ell$

$P(a) = \boxed{\ell = f(a)}$

$P'(x) = 2k(x-a) + m$, $P'(a) = \boxed{m = f'(a)}$

$P''(x) = \boxed{2k = f''(a)}$ $\boxed{k = \frac{f''(a)}{2}}$

$\boxed{f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2}$

Example 7. Find the quadratic approximation for the function $f(x) = \cos x$ near $a = 0$.

$f(x) = \cos x$, $a = 0$

$f(x) = \cos x$, $f(0) = \cos 0 = 1$

$f'(x) = -\sin x$, $f'(0) = -\sin 0 = 0$

$f''(x) = -\cos x$ $f''(0) = -\cos 0 = -1$

$\cos x \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2$

$\boxed{\cos x \approx 1 - \frac{1}{2}x^2}$

The quadratic approximation of f near a is

$$\boxed{f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2}$$

Example 8. Find the quadratic approximation to $f(x) = \sqrt[3]{x}$ near $a = -8$.

$f(x) = x^{1/3}$, $f(-8) = (-8)^{1/3} = -2$

$f'(x) = \frac{1}{3}x^{-2/3}$, $f'(-8) = \frac{1}{3}(-8)^{-2/3} = \frac{1}{3}(-2)^{-2} = \frac{1}{12}$

$f''(x) = \frac{1}{3}\left(-\frac{2}{3}\right)x^{-5/3} = -\frac{2}{9}x^{-5/3}$

$f''(-8) = -\frac{2}{9}(-8)^{-5/3} = -\frac{2}{9(-8)(-8)^{2/3}} = -\frac{2}{9(-8)(4)}$

$= \frac{1}{144}$

$f(x) \approx f(-8) + f'(-8)(x+8) + \frac{f''(-8)}{2}(x+8)^2$

$= -2 + \frac{1}{12}(x+8) + \frac{1}{2}\,\frac{1}{144}(x+8)^2$

$= \boxed{-2 + \frac{x+8}{12} + \frac{(x+8)^2}{288}}$