Chapter 3. **Derivatives**
Section 3.1 **Derivatives**

Definition. The derivative of a function f at a number a, denoted by $f'(a)$, is

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

if the limit exist.

Example 1. Find $f'(a)$ if $f(x) = \sqrt{2x - 3}$, $a > 3/2.$, $f(a) = \sqrt{2a-3}$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{(\sqrt{2x-3} - \sqrt{2a-3})(\sqrt{2x-3} + \sqrt{2a+3})}{(x-a)(\sqrt{2x-3} + \sqrt{2a+3})}$$

$$= \lim_{x \to a} \frac{(\sqrt{2x-3})^2 - (\sqrt{2a-3})^2}{(x-a)(\sqrt{2x-3} + \sqrt{2a+3})} = \lim_{x \to a} \frac{2x - 3 - (2a-3)}{(x-a)(\sqrt{2x-3} + \sqrt{2a-3})}$$

$$= \lim_{x \to a} \frac{2x - \cancel{3} - 2a + \cancel{3}}{(x-a)(\sqrt{2x-3} + \sqrt{2a-3})} = \lim_{x \to a} \frac{2\cancel{(x-a)}}{\cancel{(x-a)}(\sqrt{2x-3} + \sqrt{2a-3})}$$

$$= \lim_{x \to a} \frac{2}{\sqrt{2x-3} + \sqrt{2a-3}} = \frac{2}{\sqrt{2a-3} + \sqrt{2a-3}} = \frac{2}{2\sqrt{2a-3}} = \boxed{\frac{1}{\sqrt{2a-3}}}$$

Geometric interpretation of the derivative. $f'(a)$ is the slope of the tangent line to $y = f(x)$ at the point $(a, f(a))$.

Example 2. Find an equation of the tangent line to $f(x) = \sqrt{2x - 3}$ at the point (2,1).

Tangent line @ $(a, f(a))$: $y = f'(a)(x-a) + f(a)$

$a=2$, $f(2) = 1$, $f'(x) = \frac{1}{\sqrt{2x-3}}$ (Example 1)

$f'(2) = \frac{1}{\sqrt{2(2)-3}} = \frac{1}{1} = 1$

$y = \underbrace{1}_{f'(2)}(x-2) + \underbrace{1}_{f(2)}$

$\boxed{y = x - 1}$

Other interpretations of the derivative.

- $f'(a)$ is the instanteneous rate of change of $y = f(x)$ with respect to x when $x = a$.

- if $s = f(t)$ is the position function of a particle that moves along a straight line, then $f'(a)$ is the velocity of the particle at time $t = a$

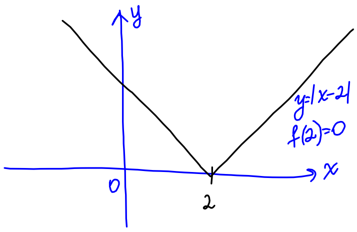
A function

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

is called the derivative of f.

Definition. A function f is differentiable at a if $f'(a)$ exists. It is differentiable on an open interval (a, b) if it is differentiable at every number in the interval.

Example 3. Where is the function $f(x) = |x - 2|$ differentiable?



$$|x-2| = \begin{cases} x-2, & \text{if } x \geq 2 \\ -(x-2), & \text{if } x < 2 \end{cases}$$

$$\lim_{x \to 2^-} \frac{f(x) - f(2)}{x-2} = \lim_{x \to 2^-} \frac{-\cancel{(x-2)}}{\cancel{x-2}} = -1$$

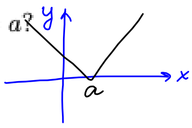
$$\lim_{x \to 2^+} \frac{f(x) - f(2)}{x-2} = \lim_{x \to 2^+} \frac{x-2}{x-2} = 1$$

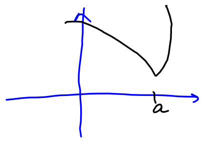
$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x-2} \quad \text{DNE}$$

$$\boxed{f(x) = |x-2| \text{ is not differentiable @ } x = 2}$$

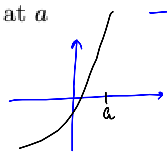
Theorem. If f is differentiable at a, then f is continuous at a

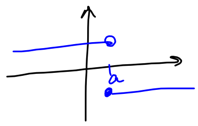
When is the function not differentiable at $x = a$?



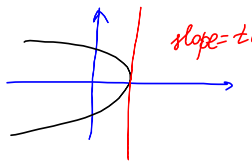


- f has a "corner" or "kink" at a





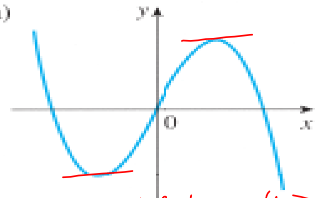
- f is discontinuous at a
- the curve $y = f(x)$ has a vertical tangent line at $x = a$



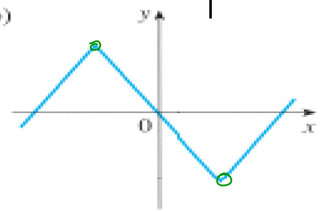
slope $= \tan\frac{\pi}{2}$ DNE

34. Match the graph of each function in (a)–(d) with the graph of its derivative in I–IV. Give reasons for your choices.

(a)

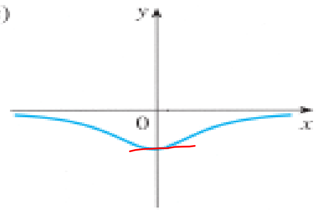


(b)

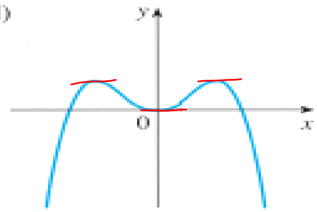


horizontal tangents $\Rightarrow f'(a)=0$

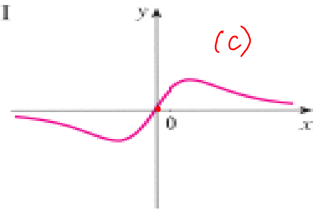
(c)



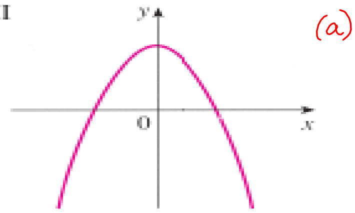
(d)



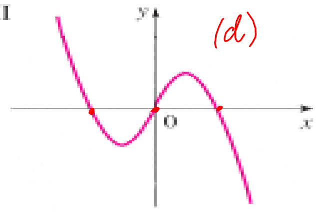
I



II



III



IV