Section 3.2 **Differentiation formulas**

Table of derivatives

1. $(C)' = 0$, C is a constant,
2. $(x)' = 1$,
3. $(x^2)' = 2x$,
4. $(x^n)' = nx^{n-1}$,
5. $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$

Differentiation formulas

Suppose c is a constant and both functions $f(x)$ and $g(x)$ are differentiable.

1. $(cf(x))' = cf'(x)$,
2. $(f(x) + g(x))' = f'(x) + g'(x)$,
3. $(f(x) - g(x))' = f'(x) - g'(x)$,
4. $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$,
5. $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$.

Find the derivative

Example 1. Differentiate each function.

(a.) $f(x) = x^5 - 4x^3 + 2x - 3$

$$f'(x) = (x^5 - 4x^3 + 2x - 3)' = (x^5)' - (4x^3)' + (2x)' - \underbrace{(3)'}_{0}$$

$$= 5x^4 - 4(x^3)' + 2(x')$$

$$= 5x^4 - 4(3x^2) + 2$$

$$= \boxed{5x^4 - 12x^2 + 2}$$

(b.) $f(x) = 3x^{2/3} - 2x^{5/2} + x^{-3}$

$$f'(x) = (3x^{2/3} - 2x^{5/2} + x^{-3})' = 3\left(\tfrac{2}{3}\right)x^{\frac{2}{3}-1} - 2\,\tfrac{5}{2}\,x^{\frac{5}{2}-1} + (-3)x^{-3-1}$$

$$= \boxed{2x^{-\frac{1}{3}} - 5x^{\frac{3}{2}} - 3x^{-4}}$$

(c.) $f(x) = (x^2\sqrt[3]{x^2}) = x^2 \cdot x^{2/3} = x^{2+\frac{2}{3}} = x^{\frac{8}{3}}$

$$f'(x) = (x^{\frac{8}{3}})' = \frac{8}{3}x^{\frac{8}{3}-1} = \boxed{\frac{8}{3}x^{\frac{5}{3}}}$$

(d.) $f(x) = \dfrac{2}{\sqrt[3]{x^2}} - \dfrac{1}{x\sqrt[3]{x}} = \dfrac{2}{x^{2/3}} - \dfrac{1}{x \cdot x^{1/3}} = \dfrac{2}{x^{2/3}} - \dfrac{1}{x^{4/3}} = 2x^{-2/3} - x^{-4/3}$

$$f'(x) = \left(2x^{-2/3} - x^{-4/3}\right)' = 2\left(-\frac{2}{3}\right)x^{-\frac{2}{3}-1} - \left(-\frac{4}{3}\right)x^{-\frac{4}{3}-1}$$

$$= \boxed{-\frac{4}{3}x^{-\frac{5}{3}} + \frac{4}{3}x^{-\frac{7}{3}}}$$

(e.) $f(x) = (x^5 + 3x^2 + 2x - 3)(x^2 + 3x + 5)$ Product Rule $[fg]' = f'g + g'f$

$$f'(x) = (x^5+3x^2+2x-3)'(x^2+3x+5) + (x^5+3x^2+2x-3)(x^2+3x+5)'$$

$$= \boxed{(5x^4+6x+2)(x^2+3x+5) + (x^5+3x^2+2x-3)(2x+3)}$$ DO NOT SIMPLIFY !!!

(f.) $g(x) = \dfrac{2x+3}{x^2 - 5x + 5}$ Quotient Rule $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - g'f}{g^2}$

$$g'(x) = \frac{(2x+3)'(x^2-5x+5) - (2x+3)(x^2-5x+5)'}{(x^2-5x+5)^2} = \frac{2(x^2-5x+5) - (2x+3)(2x-5)}{(x^2-5x+5)^2}$$

(g.) $f(z) = \dfrac{1+\sqrt{z}}{1-\sqrt{z}} = \dfrac{1+z^{1/2}}{1-z^{1/2}}$

$$f'(z) = \frac{(1+z^{1/2})'(1-z^{1/2}) - (1+z^{1/2})(1-z^{1/2})'}{(1-z^{1/2})^2} = \frac{\frac{1}{2}z^{-1/2}(1-z^{1/2}) - (1+z^{1/2})\left(-\frac{1}{2}\right)z^{-1/2}}{(1-z^{1/2})^2}$$

$$= \frac{\frac{1}{2}z^{-1/2} - \cancel{\frac{1}{2}z^{-1/2}z^{1/2}} + \frac{1}{2}z^{-1/2} + \cancel{\frac{1}{2}z^{1/2}z^{-1/2}}}{(1-z^{1/2})^2}$$

$$= \frac{z^{-1/2}}{(1-z^{1/2})^2} = \boxed{\frac{1}{z^{1/2}(1-z^{1/2})^2}}$$

Example 2. Find the equation to the tangent line to the curve $y = x + \sqrt{x}$ at the point (1,2)

$$f(x) = x + x^{1/2}, \quad f'(x) = 1 + \frac{1}{2}x^{-1/2}$$

$$f'(1) = 1 + \frac{1}{2} = \frac{3}{2}$$

Tangent line: $y = \underbrace{f'(1)}_{\frac{3}{2}}(x-1) + \underbrace{f(1)}_{2}$

$$\boxed{y = \frac{3}{2}(x-1) + 2}$$

Find the values of m and b that make $f(x)$ differentiable everywhere.

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 7 \\ mx + b & \text{if } x > 7 \end{cases} \quad , \qquad f'(x) = \begin{cases} 2x, & \text{if } x \leq 7 \\ m, & \text{if } x > 7 \end{cases}$$

[redacted]

- $f'(x)$ must be continuous everywhere

$$\lim_{x \to 7^-} f'(x) = \lim_{x \to 7^+} f'(x)$$

$$\left.\begin{array}{l} \lim_{x \to 7^-} f'(x) = \lim_{x \to 7} (2x) = 14 \\ \lim_{x \to 7^+} f'(x) = \lim_{x \to 7^+} m = m \end{array}\right\} \quad \boxed{m = 14}$$

$f(x)$ must be continuous everywhere

$$\lim_{x \to 7^-} f(x) = \lim_{x \to 7^-} (x^2) = 49 \; ; \; \lim_{x \to 7^+} f(x) = \lim_{x \to 7^+} (mx + b) = 7m + b$$

$$49 = 7m + b \Rightarrow b = 49 - 7m$$
$$= 49 - 7(14)$$
$$\boxed{b = -49}$$

Find equations of both lines through the point (2, -3) that are tangent to the parabola $y = x^2 + x$.